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THE INTERNATIONAL JOURNAL OF TECHNOLOGY IN MATHEMATICS EDUCATION

Volume 11, Number 1

Contents

Editorial

Welcome to the first issue of the International Journal of Technology in Mathematics Education. The title of the journal has been changed to broaden the scope of the journal. While articles on the use of computer algebra systems will continue to be a key component of the journal's content, it hoped to encourage more papers on the use of other technologies. In particular papers on other mathematical computer applications and also hand held technology, such as graphics calculators will be encouraged. The change in title has been as a response to the number of papers that have been submitted to the journal that could not strictly be described as computer algebra. We hope that the readers will find that the wider scope encourages a greater variety of papers, that will be of interest to a wider audience.

There has also been a change of editor with this edition of the journal. John Berry has worked as the editor for ten volumes of the journal since its original conception as the International Derive Journal in 1994. I would like to thank John for his leadership over this period and for his hard work and commitment to the production of a quality journal. John will continue to be involved as an assistant editor.

The final change that you will observe with this issue is that the format has changed. We hope that the new format will allow us to present the contents of the paper in a more coherent way, with more information on each page.

This first issue contains two research papers. The first of these, by Pierce and Stacey, sets out a theoretical framework for research into the use of CAS in teaching and learning. The development of theoretical frameworks is vital to underpin research studies in all areas, but new technologies seem to pose new challenges and often require more sophisticated models. The work in this paper concentrates on the idea of "algebraic insight". The authors present their framework and then illustrate how it can be applied in regard to algebraic expectation.

The second research paper, by Forster, looks at the use of graphics calculators by students and in this sense illustrates the reason for the change of title of the journal. Her emphasis is on how students learn to use graphics calculators efficiently. The research is based on observations of classes using a graphics calculator regularly and in particular on any discussion that took place as to how to use them. As a result of this work the author is able to make recommendations to teachers, to enable them to help their students become more efficient users of their graphics calculators.

The journal also contains two Teaching and Learning papers. The first of these, by Zand and Crowe, tackles the issue of how CAS can be used to help students whose skills in algebraic manipulation are not strong. This issue has been of concern in a number of areas and the authors use an example taken from a vector calculus course.

Finally we include an article by Sriraman and Strzelecki. This is an interesting paper that does not specifically make use of CAS or other computer packages. The authors have set out to show students some of the limitations of using computational tools and the power of a mathematical argument. We have included this article because it provides examples that can be investigated with some forms of technology, but for which mathematical reasoning may be more appropriate. In their conclusion, the authors discuss how their problems require the students to look for alternatives when they encounter the limitations of the technology that they are working with.

I hope that you enjoy reading this first issue of the new journal.

Ted Graham

July 2004

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Monitoring Progress in Algebra in a CAS Active Context: Symbol Sense, Algebraic Insight and Algebraic Expectation

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The purpose of this paper is to provide researchers with a shared framework, terminology and tool to improve the coherence of research into learning mathematics with CAS and to assist its findings to accumulate into a significant body of knowledge. Experience with calculators in arithmetic led to a framework for number sense. There is an obvious parallel for algebra, where the development of algebraic insight to monitor symbolic work will assume high importance. We present a framework for algebraic insight then explore one aspect, algebraic expectation, in detail. Just as estimation is a valued skill for monitoring arithmetic calculations, we suggest that expectation should be a focus in teaching algebra, especially when symbolic technology is available. Through typical examples, we demonstrate the value of the algebraic insight framework for monitoring students' work with CAS.

KEYWORDS
algebra, algebraic expectation, computer algebra systems, number sense, symbol sense.

1. INTRODUCTION

Faced with the increasing availability of Computer Algebra Systems (CAS) for doing, teaching and learning mathematics, both teachers and researchers question what algebra should be taught. They fear that students will merely replace the memorisation of algebraic manipulation routines with the memorisation of calculator specific button sequences. In a CAS active context, what aspects of algebra are important and what aspects should we be monitoring in order to judge a student's progression in developing facility with algebra? We propose that the answer to this question is the construct we call 'algebraic insight' and we have organised the components of this construct in a framework that is useful as a guide to both teachers and researchers monitoring students' progress. Pierce (2002) reported the details of a study which provided the context for the development of this framework and in a related paper, Pierce and Stacey (2002), we described the place of this construct when teaching algebra. In this paper we emphasise the need for such a framework for research, explain the components of algebraic insight and then illustrate the use of the framework in monitoring individual student's algebra progress. First we situate our thinking in terms of three key papers from the 1990s related to this topic.

2. NUMBER SENSE AND SYMBOL SENSE

The last twenty-five years have seen the increasing availability and affordability of technology that will carry out routine mathematical processes in educational settings: first arithmetic calculators and now, graphical calculators and symbolic manipulators. Once students became able to use four function calculators to do arithmetic, researchers and educators saw more clearly than before, that there is more to arithmetic than calculation. Now symbolic manipulators, in the form of Computer Algebra Systems, are doing the same for algebra.

After a decade or more of experience of teaching with four function calculators, the consensus amongst educators was that a new emphasis for arithmetic could be placed on the understanding and ability to plan, monitor, estimate and interpret arithmetic calculations. This ability, described by the term number sense is well summarised by McIntosh, Reys and Reys (1992) who gave a comprehensive definition of number sense in a framework (see Appendix 1) that organised its component parts. McIntosh et al said that this framework was not an exhaustive listing of all possible components of number sense (an impossible task) but 'an attempt to articulate a structure which clarifies, organises, and interrelates some of the generally agreed upon components of number sense' (p5). We aim to put forward an equivalent framework for algebra.

While students working with the three representations of functions offered by CAS still need number sense, they also require equivalent abilities for working with algebraic symbols and graphs which, by analogy, have been called symbol sense and graph sense. There have been two important attempts to describe symbol sense, by Fey (1990) and Arcavi (1994). First, Fey (1990), who was reflecting on the impact of technology on teaching and learning mathematics, suggested five basic abilities (see Appendix 2) that are each part of the thinking that enables a mathematician to recognise equivalent expressions or form an expectation of the nature of the result of a problem. The construct put forward in this paper as algebraic insight encompasses the abilities touched on by Fey (1990) but expands these to a more comprehensive set. Second, Arcavi (1994) proposed a very general interpretation of symbol sense (see Appendix 3), which applies across the problem solving process when formulating a problem algebraically, solving the mathematically formulated problem and interpreting the results in terms of the original problem. More recently Boero (2001) has described algebraic anticipation, the ability not only to apply standard patterns of transformations but also to

> …foresee some aspects of the final shape of the object to be transformed related to the goal to be reached, and some possibilities of transformation. This 'anticipation' allows planning and continuous feedback. In the case of transformations after formalisation, anticipation is based on some peculiar properties of the external algebraic expression. (p99).

Boero's algebraic anticipation is also related to both symbol sense and algebraic insight, and appears to be similar to the concept described below as algebraic expectation.

Algebraic insight impacts on 'solving' symbolically formulated problems.

Figure 1 illustrates where algebraic insight is located with respect to Arcavi's more general 'symbol sense', in terms of a basic model of the problem solving process. Symbol sense is involved in the formulating, solving and interpreting stages. Algebraic insight, designed to capture the insight that a student needs to work with algebraic symbols in transformational activity (Kieran, 1996), is relevant only in the "solving" phase of the problem solving process.

Figure 1 also shows that algebraic insight is relevant whether or not one is using CAS, but it is the CAS environment that has highlighted its importance and in turn, it is a concept that is especially relevant for studying students' progress when using CAS. Many of the symbol sense abilities described by Arcavi (1994) are unaffected by the availability of CAS since CAS only performs manipulations and calculations facilitated by algorithmic routines; it does not set up a model, nor decide how best to solve a problem; nor does it interpret results.

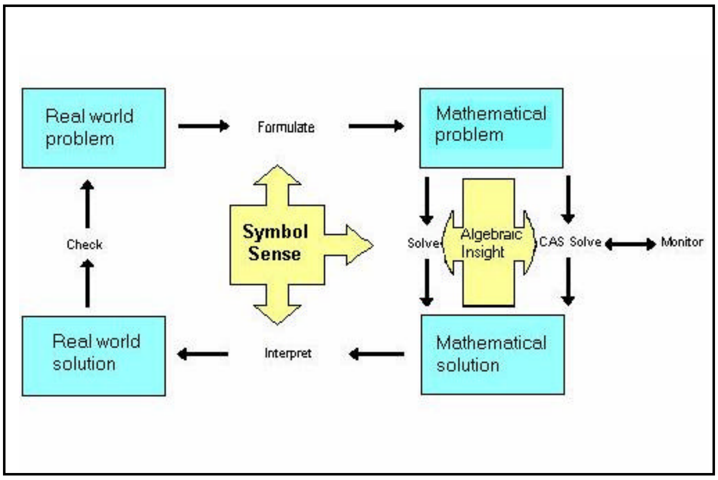

Figure 1. A model of problem solving showing the places of symbol sense and algebraic insight (Pierce & Stacey , 2002).

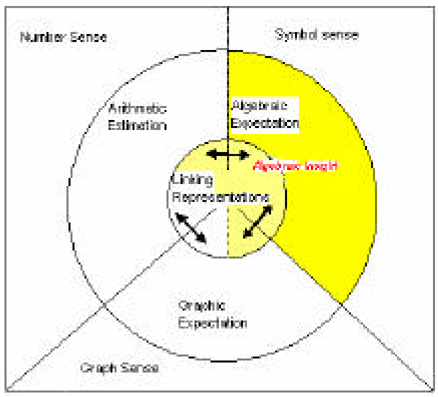

Figure 2. The place of algebraic insight and its components within the senses needed when working with CAS.

Figure 2 demonstrates the relationships between the various constructs in this paper from a set-theoretic viewpoint. The whole picture is divided into three sections: number sense, graph sense and symbol sense (as described by Arcavi) relevant to dealing with algebraic work in symbolic, graphical and numeric representations. The right hand side of figure 2 indicates that algebraic insight is mostly but not entirely within symbol sense and that it is divided into two parts, algebraic expectation and the ability to link the symbolic to other representations, which will be described below.

The framework presented below organises and exemplifies these constructs; then its use as a guide for analysing students' work and hence monitoring their progress is explored through typical examples observed when students work with CAS. The framework we propose is based on both our experience of students and that reported in other studies. The next section emphasises the need for such a framework in research.

3. THE NEED FOR AN AGREED FRAMEWORK

One of the purposes of proposing and delineating the construct of algebraic insight is for use when researching the links between algebraic knowledge and the use of CAS, especially its symbolic manipulation facility. Algebraic insight is a useful concept to guide teaching (as will be illustrated in our examples below) but our main intention for this paper is that a careful delineation will provide researchers with a shared framework, terminology and tool to improve the coherence of research and to assist its findings to accumulate into a significant body of knowledge.

In this, we are motivated by recent proposals that education researchers should more frequently use the same variables, theoretical constructs and measures across different research studies. The USA National Research Council's Committee on Scientific Principles for Educational Research (Shavelson and Towne, 2002) has identified this as a priority to support the accumulation of research-based knowledge in education. The RAND Mathematics Study Panel (2003) goes further when it proposes that US government agencies might consider enhancing the infrastructure for research and development by "a special effort to assemble and, where necessary, develop measurement instruments and technology that could be widely used by researchers, and thus enhance the opportunities for comparing and contrasting findings of various research efforts" (p 70). Making a similar point, Burkhardt and Schoenfeld (2003) call for "a requirement to justify not using established instruments and methods" in order to increase comparability of studies. Shavelson and Towne, RAND, and Burkhardt and Schoenfeld all call for shared instruments as well as shared constructs. Space limitations preclude presentation of instruments to assess algebraic insight here, but suggestions with reports of results from students can be found in Ball, Stacey and Pierce (2001), Pierce (2002) and Ball, Pierce and Stacey (2003).

4. A FOCUS WITHIN SYMBOL SENSE: ALGEBRAIC INSIGHT

We see competence in algebraic insight as having two aspects: algebraic expectation and ability to link representations. Since the use of graphical calculators has already raised much discussion of the thinking involved in linking representations (see for example Dick, 1992) its place as part of algebraic insight is established in the framework but it is not discussed in depth in this paper where we focus instead on describing algebraic expectation: the thinking that must accompany the formal symbolic operations and transformational activity of algebra even when CAS is available.

The term algebraic expectation is used here to name the thinking process that takes place when an experienced mathematician considers the nature of the result they expect to obtain as the outcome of some algebraic (and symbolic) process. For example, it takes place when a mathematician looks at two expressions and decides, without doing any explicit calculations or manipulations, whether they are likely to be equivalent.

Algebraic expectation does not necessarily involve producing an approximate solution as in arithmetic “estimation” (there are differences in what can be done mentally with algebra and arithmetic) but rather noticing these ‘peculiar properties’, conventions, structure, and key features of an expression that determine features which may be expected in the solution. Students need to develop skill with scanning expressions for these clues that allow us to see and predict patterns, and make sense of symbolic operations.

Aspects	Elements	Common Instances
1. Algebraic Expectation	1.1 Recognition of conventions and basic properties	1.1.1 Know meaning of symbols 1.1.2 Know order of operations 1.1.3 Know properties of operations
	1.2 Identification of structure	1.2.1 Identify objects 1.2.2 Identify strategic groups of components 1.2.3 Recognise simple factors
	1.3 Identification of key features	1.3.1 Identify form 1.3.2 Identify dominant term 1.3.3 Link form with solution type
2. Ability to Link representations	2.1 Linking of symbolic and graphic representations	2.1.1 Link form with shape 2.1.2 Link key features with likely position 2.1.3 Link key features with intercepts and asymptotes
	2.2 Linking of symbolic and numeric representations	2.2.1 Link number patterns or type with form 2.2.2 Link key features with suitable increment for table 2.2.3 Link key features with critical intervals of table

Table 1. A Framework for algebraic insight

5. A FRAMEWORK FOR ALGEBRAIC INSIGHT

The framework presented in Table 1 aims to articulate a structure that clarifies, organises, and interrelates key elements of algebraic insight. It is not proposed as a catalogue of specific, itemised skills. The divisions within the framework are neither mutually exclusive nor exhaustive. It is an attempt to analyse what it is that 'expert' mathematicians do when they look at a result for an algebraic problem and say 'there is a mistake here' or 'that looks OK'. This is the thinking used in what the problem solving literature (see for example, Schoenfeld, 1985) calls 'monitoring' or 'control'. Mathematics teaching has, of necessity, focused a great deal of time and attention on algorithmic routines. Since CAS does these effectively, perhaps attention can be directed towards deliberately teaching these skills of algebraic insight.

As is shown in Figure 2, and column 1 of Table 1, two aspects, algebraic expectation and ability to link representations, form a logical division of algebraic insight. Algebraic expectation focuses on the application of algebraic insight within the symbolic representation of mathematics. A simple example is 'knowing to expect 0, 1 or 2 real solutions to a quadratic equation'. The ability to link representations deals with the students' ability to move cognitively between symbolic (algebraic) representations and graphical or numeric representations. Such linking is also concerned with expectations, but expectations across representations. Algebraic insight will be shown when a student has expectations about graphs and tables that are linked to features of the symbolic representation; for example when a student asks and answers such questions as:

> What will the graph of the rule $y = x^2 + 5$ look like? Should I expect it to cross the x-axis? If I am to construct a table of values for this rule, what might be a suitable increment to use?

The framework suggests three key elements of algebraic estimation and two key elements of ability to link representations. These elements are shown in column 2 of Table 1 and each element is then illustrated in column 3 by typical common instances. Such common instances of the elements of each aspect may be seen when students demonstrate the abilities listed in the third column. The column 3 abilities do not form a definitive list but were selected from examples observed by the first author while teaching a functions and introductory calculus course for undergraduate students. In general, while the aspects and the elements of algebraic insight apply at any level, details of common instances will be specific to both age and stage.

6. ALGEBRAIC INSIGHT – WHAT CAN WE LEARN FROM STUDENTS WORKING WITH CAS?

Algebraic insight, although important whether working by-hand or with technology, is brought into sharper focus when a CAS is available to perform routine processes. Observing and measuring students' algebraic insight is therefore an appropriate way to monitor students' progress in algebra when working in a CAS active context. We therefore illustrate our choice of key aspects and elements for algebraic insight, through examples of 'typical' students working with CAS in a functions and introductory calculus course. Again, because the ability to link representations has had considerable prominence in research on the use of graphical calculators, we have chosen examples that highlight algebraic expectation.

The action of using the symbolic module of a CAS to perform formal symbolic operations can be divided into four stages: first choosing whether to use CAS or not, second keying in or entering an expression, third monitoring the solution processes and finally confirming the solution(s). While it is possible that a student may use CAS to obtain correct solutions to some problems by merely pushing buttons in a prescribed or memorised sequence, constructive progress in mathematical understanding and the application of techniques to new problems will be facilitated when effective use of CAS (Pierce & Stacey, in press) is accompanied by well developed algebraic expectation.

Algebraic Expectation informs decisions about when to use CAS

Faced with a symbolically formulated problem, a student must make decisions about likely solution paths. If CAS is available one of the choices to be made is whether to use it or not. Just because it is available does not mean that its facility offers the most efficient means of solving a problem. Learning to use CAS effectively includes learning to make judicious choices about its use (Pierce & Stacey, in press.).

Students typically decide to use CAS for problems that they expect will be difficult or time-consuming (Pierce, 2002). Identification of objects (Framework code 1.2.1), recognition of simple factors (1.2.3) and identifying form (1.3.1) are common instances of algebraic expectation fundamental to such judgements. For example a student who, on reading $f(x) = \sin^2(2x+1) + \cos^2(2x+1)$, identified $2x + 1$ as an object and recognised the form of the trigonometric identity, therefore realising that $f(x) = 1$, would hardly proceed by entering the original form of the function into CAS. Similarly a student recognising $(x + 1)$ as an object and simple common factor would most likely simplify the expression $\dfrac{a(x+1)^5 + b(x+1)^2}{x+1}$ by hand since this would be faster than entering this long expression into a CAS. Entering expressions carries the risk of typographical and syntax errors along with a need for algebraic expectation, as illustrated by observing a typical student like Alice described below.

Tall and Thomas (1991) emphasised the importance of a structural view of algebra, and drew attention to the dual role of algebra expressions as both processes and objects. An important example of structure (1.2) occurs when composite functions, like those above, are used. Working with CAS is often greatly facilitated by adopting a structural view, so it supports and sometimes demands their use. For example, at the outset of a problem, it is often very helpful to define (or name) a function and use the name in further processes, for example: $f(g(x))$ if $g(x) = x + h$ and $f(x) = 5x^3 - 2x^2 + x + 7$. To show algebraic expectation a student would not need to perform the expansion of this new expression, but identify ($x + h$) as an object that will replace each x. Students often make errors when creating such composite functions (for example, only replacing the first x of an expression with the new $f(x)$ object) and so may be glad to assign the task of simplification to CAS. This allows us to monitor a deeper level of students' understanding that may have been masked by simple procedural errors, which, while not unimportant, often result from a lack of attention to detail or perhaps test anxiety. In any case many standard tests and classic items that test these procedural skills have already been developed and results of these studies are widely known.

Algebraic expectation is needed when entering expressions into CAS: Alice

When working by-hand a student can copy a question from a textbook but may make no progress towards the solution. It is often difficult to know what is blocking that student's progress but, sometimes, when they use CAS, their difficulties are more apparent to a teacher. Even in the initial stage of entering expressions the obstacles that students encounter are sometimes easier to identify when they are working with CAS. Apparently trivial errors, such as omitting a bracket, can often reveal fundamental mathematical difficulties as evidenced by the example of Alice's work, described in figure 3.

Alice was bewildered when she entered expressions into CAS and the resulting screen image did not match the printed or handwritten version in front of her. In a first instance, her error of omitting parentheses is not unexpected since as Booth (1988) noted, when working by hand, younger or less able students tend to ignore parentheses, evaluating everything from left to right. In order to enter such an expression correctly, Alice needed to first identify the structure of the expression (Framework code 1.2.2) to recognise this as a ratio of two functions ($x^2 + 1$) and ($2 - x$) which must be made explicit for the machine. In addition, to understand the mismatch with the CAS output from her entry line, Alice needed to know that in mathematics there is an agreed convention for the order of operations (1.1.2) and that she should expect a CAS to be programmed to follow that convention. Thus, in the absence of suitably placed parentheses, her CAS performed the multiplications and divisions indicated before the additions and subtractions. When Alice had corrected her syntax she then needed to know the properties of operations (1.1.3) in order to recognise the equivalence of the two expressions $\frac{x^2+1}{2-x}$ and $\frac{-(x^2+1)}{x-2}$.

When working with the function $f(x) = \frac{x^2+1}{2-x}$ Alice typed the sequence x^2+1/2-x, resulting in the screen image $x^2 + \frac{1}{2} - x$. She was bewildered.

Alice re-entered the expression using correct syntax and then wanted to check that she had been successful in correctly defining this function so she typed f(x) again in the entry line and pressed the enter key. Again, to her surprise the expression which appeared on the screen was not $\frac{x^2+1}{2-x}$ but $\frac{-(x^2+1)}{x-2}$.

Later, when working on a variation of this first problem, Alice correctly entered the expression $\frac{x^2-1}{x-1}$ as (x^2-1)/(x-1). The CAS showed the expression as she had hoped on the left of her screen but, on the right, just showed x+1. Again she was surprised.

Figure 3. Alice needs to recognise equivalent expressions

The variation on the first problem produced a further disturbing output for Alice. She needed to recognise these as equivalent expressions (1.1.3), provided $x \neq 1$, through recognition of simple factors (1.2.3) in the difference of two squares.

A new problem required Alice to use the expression $ae^x \sin px$. As she worked on this problem, the designated 'e^x', 'π' and 'sin' buttons reminded her that letters can also be used to represent specific values and to name processes (1.1.1). Even the separate X key, accessed without using the 'alpha' mode reinforced the common convention that x would represent a variable while a would be more likely to represent a constant value in that problem.

Mathematics teachers know that the meaning students assign to letter symbols is fundamental to their use of algebra. Many researchers, since the work of Küchemann (1981), have shown that there is a great deal of variety in these meanings and that a great deal of misunderstanding results from assigning inappropriate meaning to symbols. To show algebraic insight students must understand that letters can be used with different meanings in different contexts and be able to recognise which meaning is applicable for a particular expression. Working with CAS has helped to develop Alice's knowledge of the meaning of symbols (1.1.1).

Algebraic expectation applied when monitoring symbolic processes: Ben and Craig

Students need strategies for monitoring their work, and these are an essential part of algebraic expectation. First they need to be able to identify the inevitable typographical or keying errors which occur when entering an expression or function into CAS. Next they must monitor the algebraic processes and routines they choose to initiate through the use of CAS commands.

Ben and Craig have been asked to factorise the quartic expression $x^4 - 5x^2 + 4$. They have previous experience only with quadratics and cubic polynomials. Before entering anything, Ben predicted that there would be four factors and then they set to work, each using a CAS calculator.

Ben: "There will be four factors" (predicting before entering)

Both boys enter the expression on their separate machines and select the "factorise" command.

Ben's calculator shows: $(x-1)(x+1)(x-2)(x+2)$

Ben: "That's what I expected, 4 factors"

Craig's calculator shows:

$$\left(x^2 + \frac{\sqrt{13}x}{2} + 1\right)\left(x^2 - \frac{\sqrt{13}x}{2} + 1\right).4$$

Craig (surprised): "Oh wow! How come I got that? I expected an answer like your's - not mine- because with quadratics it's two factors, with cubics it's three, therefore with that pattern with quartics it's four!"

Ben: (musing on why Craig's result might be correct even though there are only two factors) "If you look [at the expression] x^2 times x^2 is x^4"

Craig: "Yes, but it's not in simplest form, factorising simplifies."

Ben: "We could expand your expression to see if they are both correct."

Ben's expansion returned the original expression

$x^4 - 5x^2 + 4$

Craig's expansion returned the expression $4x^4 - 5x^2 + 4$.

This input error explained the discrepancy, although there is more for them to learn.

Figure 4. Ben and Craig demonstrate algebraic expectation

Ben and Craig, for example were working together to factorise polynomials. They had had experience with quadratics and cubics but this was the first time they were faced with a quartic expression, $x^4 - 5x^2 + 4$. A scenario of their working is set out in Figure 4. Here we see algebraic expectation at work. Ben and Craig identified two key features of the expression; it was a polynomial (1.3.1) and the dominant term was x^4 (1.3.2). This informed their expectation that there were likely to be four factors. Their previous experience also led them to believe that the factors would be linear. In addition, Ben showed knowledge of the properties of operations (1.1.3) when he suggested that expanding Craig's result could help resolve the discrepancy. Expansion of Ben's result returned the original expression, whereas expansion of Craig's returned the expression $4x^4-5x^2 + 4$. Neither of the boys, nor the two researchers present, had noticed the 4 on the far right of Craig's factorisation, which would have given them an easy clue to the explanation of the different factorisations. This illustrates how the limitations of current calculator screen size place additional demands on students' algebraic expectation.

Algebraic expectation supports confirmation of solutions

Linking form to solutions will inform expectation about the possible minimum and maximum number of solutions, the nature of solutions to be expected and the range over which a function may be defined. Identifying form for functions can mean noting that $x^4 + 4x^3 + 6x^2 + 4x + 1$ is a polynomial of degree 4, $2 + e^x$ is exponential and $4\sin 3x$ is trigonometric. At a second level it can mean seeing the symmetry of the coefficients of $x^4 + 4x^3 + 6x^2 + 4x + 1$ with descending powers of x suggesting a binomial expansion and that $e^{2x} + e^x - 2$ is a quadratic in e^x. Identifying form provides checks on the equivalence of two different expressions of solutions and whether all likely solutions to a problem have been found.

7. CONCLUSION

The illustrations outlined above make it clear that the availability of CAS does not mask students' understanding of algebra. Students' progress will be seen in their improved ability to choose appropriate routines and monitor their progress towards solutions. Such progression requires the development of algebraic insight.

Conceptualising the symbol sense required in the solving phase of problem solving (Figure 1) as consisting of the ability to link representations and algebraic expectation provides a framework for studying the likely effects of CAS use on curriculum, teaching and learning. Finally, the notion of algebraic expectation seems to capture particularly well the algebraic skill that parallels arithmetic estimation, and should thus not only be a useful focus for teaching but also a guide for monitoring students' progress.

We hope that the framework for algebraic expectation, presented in this paper as the insight to accompany formal symbolic operations, may provide a shared structure for curriculum planning, monitoring student progress and planning research.

REFERENCES

Arcavi, A. (1994). 'Symbol sense: informal sense-making in formal mathematics'. For the Learning of Mathematics, **14 No 3**, 24-35.

Ball, L., Pierce, R. & Stacey, K. (2003). Recognising Equivalent Algebraic Expressions: An important component of algebraic expectation for working with CAS. In N. A. Pateman, B. J. Dougherty and J.T. Zillox (Eds.) Proceedings of the 27th Annual Conference of the International Group for the Psychology of Mathematics Education, Hawaii July 2003 Volume 4, 15-22. Hawaii: CRDG, College of Education, University of Hawaii.

Ball, L., Stacey, K., & Pierce, R. (2001). Assessing Algebraic Expectation. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.) Numeracy and Beyond: Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia, Volume 1, 66-73. Sydney: Mathematics Education Research Group of Australasia.

Boero, P. (2001). Transformation and anticipation as key processes in algebraic problem solving. In R. Sutherland, T. Rojano, A. Bell and R.Lins (Eds.) Perspective on School Algebra. pp. 99-120. Dordrecht, Netherlands: Kluwer Academic Publishers.

Booth, L. (1988). 'Children's difficulties in beginning algebra'. In A. F. Coxford and A. P. Shulte (Eds.), The Ideas of Algebra, K-12, 1988 Yearbook, pp 20-32. Reston, U.S.A.: National Council of Mathematics Teachers.

Burkhardt, H. and Schoenfeld, A. (2003). Improving educational research: Toward a more useful, more influential and better funded enterprise. Educational Researcher. **32 No 9** 3-14.

Dick, T. (1992). 'Super calculators: implications for calculus curriculum, instruction and assessment', in J.T. Fey and C.R. Hirsch (Eds.), Calculators in Mathematics Education, 1992 Yearbook, Reston, U.S.A.: National Council of Mathematics Teachers, 145-157.

Fey, J. T. (1990). 'Quantity', in L. A. Steen (Ed.), On the Shoulders of Giants: New Approaches to Numeracy, Washington: National Academy Press, 61-94.

Heid, M. K. (1988).'Resequencing skills and concepts in applied calculus using the computer as a tool', Journal for Research in Mathematics Education, **19 No 1**, 3-25.

Kieran, C. (1996). 'The changing face of school algebra', in C. Alsina, J.M.Alvarez, B.Hodgson, C.Laborde and A.Perez (Eds.), 8th International Congress on Mathmeatical Education. Selected Lectures, Seville, Spain: S.A.E.M. Thales, 271-290.

Küchemann, D. (1981). 'Algebra', in K. M. Hart, (Ed.), Children's Understanding of Mathematics: 11-16, Oxford, U.K.: John Murray, 102-119.

Lagrange, J.-B. (1999). 'Complex calculators in the classroom: theoretical and practical reflections on teaching pre-calculus', International Journal of Computers for Mathematical Learning**, 6 No 4**, 51-81.

McIntosh, A., Reys, B. J. and Reys, R. E. (1992). 'A proposed framework for examining basic number sense', For the Learning of Mathematics, **12 No 3**, 2-8.

Palmiter, J. (1991). 'Effects of computer algebra systems on concept and skill acquisition in calculus', Journal of Research in Mathematics Education, **22 No 2**, 151-156.

Pierce, R. (2002). An exploration of algebraic insight and effective use of computer algebra systems. Ph.D. thesis. University of Melbourne. Available: http://adt1.lib.unimelb.edu.au/adt-root/public/adt-VU2003.0004/index.html

Pierce, R. & Stacey, K. (2002). Algebraic Insight: the algebra needed to use CAS The Mathematics Teacher, **95 No 8**, 622-627.

Pierce, R. & Stacey. K., (in press). A framework for monitoring progress and planning teaching towards effective use of computer algebra systems. International Journal of Computers for Mathematical Learning.

RAND Mathematics Study Panel (2003). Mathematical proficiency for all students. Santa Monica, Ca: RAND.

Schoenfeld, A. H. (1985). Mathematical Problem Solving, Orlando: Academic Press.

Shavelson, R. & Towne, L. (2002). Scientific Research in Education. (Report of Committee on Scientific Principles for Education Research). Washington, D.C.: National Academy Press.

Tall, D. and Thomas, M. (1991). 'Encouraging versatile thinking in algebra using the computer', Educational Studies in Mathematics, 22, 125-147.

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APPENDIX 1

Framework for considering number sense (first two columns only). (McIntosh et al, 1992, p4)

1	Knowledge of and facility with NUMBERS	1.1	Sense of orderliness of numbers
		1.2	Multiple representations of numbers
		1.3	Sense of relative and absolute magnitude of numbers
		1.4	System of benchmarks
2	Knowledge and facility with OPERATIONS	2.1	Understanding the effect of operations
		2.2	Understanding mathematical properties
		2.3	Understanding the relationship between operations
3	Applying knowledge of and facility with numbers and operations to COMPUTATIONAL SETTINGS	3.1	Understanding the relationship between problem context and the necessary computation
		3.2	Awareness that multiple strategies exist
		3.3	Inclination to utilize an efficient representation and/or method
		3.4	Inclination to review data and result for sensibility

APPENDIX 2

Fey's (1990) basic components of symbol sense

F1 Ability to scan an algebraic expression to make rough estimates of the patterns that would emerge or graphic representation …

F2 Ability to make informed comparisons of order of magnitude for functions with rules of the form n_1, n_2, n_3,. n_k…

F3 Ability to scan a table of function values or a graph or to interpret verbally stated conditions, to identify the likely form of an algebraic rule that expresses the appropriate pattern…

F4 Ability to inspect algebraic operations and predict the form of the result, or as in arithmetic estimation, to inspect the result and judge the likelihood that it has been performed correctly…

F5 Ability to determine which of several equivalent forms might be most appropriate for answering particular questions… (Fey, 1990, pp 80-81 numbering added)

APPENDIX 3

Arcavi's (1994) summary of symbol sense

A1 An understanding of and aesthetic feel for the power of symbols: understanding how and when symbols can and should be used in order to display relationships, generalisations, and proofs which are otherwise hidden and invisible.

A2 A feeling for when to abandon symbols in favour of other approaches in order to make progress with a problem, or in order to find an easier or more elegant solution or representation.

A3 An ability to manipulate and to 'read' symbolic expressions as two complementary aspects of solving algebraic problems. Detached from the meaning or context of the problem and with the symbolic expression viewed globally, symbol handling can be relatively quick and efficient. On the other hand, the reading of the symbolic expressions towards meaning can add layers of connections and reasonableness to the results.

A4 The awareness that one can successfully engineer symbolic relationships that express the verbal or graphical information needed to make progress in a problem, and the ability to engineer those expressions.

A5 The ability to select a possible symbolic representation of a problem, and, if necessary, to have the courage, first, to recognise and heed one's dissatisfaction with that choice, and second, to be resourceful in searching for a better one as replacement.

A6 The realisation of the constant need to check symbol meanings while solving a problem, and to compare and contrast those meanings with one's own intuitions or with the expected outcome of the problem.

A7 Sensing the different 'roles' symbols can play in different contexts.

(Arcavi, 1994, p31, numbering added)

Efficient Use of Graphics Calculators in High School Calculus

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This paper provides a pragmatic view of efficient use of graphics calculators. Efficiency is described in terms of quick and easy calculation, as debated and evidenced in a Year 12 calculus class. Students' methods of calculation are analysed in terms of the algebraic understanding and technical skills that underpinned them. Patterns in students' selection of methods from the available possibilities and the social promotion of methods are described. I argue that students showed critical attitudes towards quickness in calculation, that some were innovative and performed advanced algebra in their pursuit of quickness, and I identify implications of the students' actions for teaching and learning.

1. INTRODUCTION

In this paper, I discuss quick and easy methods of calculation on a graphics calculator as debated and decided by the teacher and students in a Year 12 calculus class. The 17-year old students were in their final year of secondary schooling and were preparing for the Western Australian university entrance examinations. The examinations provided a reason for quick and easy methods to be mastered. Each student owned a Hewlett Packard HP38G calculator. Graphics calculators without computer algebra systems and the HP38G, which has limited algebra capabilities, are allowed for the examinations.

In the paper, I describe the methods of calculation that were accorded quick and easy status by the teacher and students and discuss the practical bases of the quickness. As well, I discuss the understanding that the approaches drew on, focussing on algebraic understanding. I finish by describing the social means through which quickness and ease were encouraged and decided.

The students' approaches to calculation on their calculators partly reflected Artigue's (1996) findings on efficient use of full computer algebra systems. Quickness and ease were pursued (a) in numerical calculation, (b) through selection of algebraic solutions in preference to graphical solutions and vice versa, and (c) involved co-ordination of numerical, graphical and algebraic work. The social means to quickness, in the main, comprised debate, direction by the teacher, and use of a view screen with students' calculators attached.

Potential points of interest in the paper, for the reader, will be students' use of tables, graphs, and commands on the calculator; the ways the use changed over time; the transportation of information between different platforms of the calculator; and the nature of activity when the view-screen was used. The paper follows others on open-ended questioning by the teacher (Davis and Forster, 2003) and the students' wit and communicative competence (Forster et al., 2002; Forster and Taylor, 2003).

2. CONTEXT

The larger study of which this paper is part resembled a study of exemplary practice (Fraser and Tobin, 1989). I invited the teacher to participate because I knew from his professional development activities for other teachers that he was active in using graphics calculators in his teaching; and knew that many of his past students had achieved outstanding results in tertiary entrance examinations. The study was non-interventionary in that I did not provide or suggest activities.

The school was a private college for girls. It had primary and secondary departments. Some of the 13 students in the calculus class had attended the college in the primary years. Most had entered the college in Year 8, the first year of secondary schooling and some had started in Year 11, which is the first year of the two-year university entrance courses.

The university entrance examinations directly test Year 12 work only. Examinations are offered in Discrete Mathematics, Applicable Mathematics and Calculus, which attract about 6000, 5000 and 2000 candidates respectively. The Discrete Mathematics and Applicable Mathematics syllabi overlap, with Discrete Mathematics being mathematically less-demanding. Students who study Calculus generally also study Applicable Mathematics. Calculus is the most demanding subject.

Graphics calculators are mandated for all three courses and examination questions can require their use. Usually, there are no instructions directing students to use the calculators in the calculus examination but rather, in accordance with the syllabus (Curriculum Council, 2003), students are expected to use the technology appropriately. Analyses of the role of the calculators in the calculus examinations for 1998-2000 are provided in Forster and Mueller (2001, 2002).

3. RESEARCH METHOD

I attended 21 fifty minute lessons in the Year 12 class. Nine were on applications of calculus (area, volume, rectilinear motion, exponential growth and decay) and 12 were on the vector calculus. I observed whole-class work, was an assistant teacher during individual and small-group work, and made field notes in both these roles. When issues emerged about calculator use, I discussed them informally with the teacher after class and made field notes on the discussions. Students' one-to-one conversations were audio-recorded and the recordings also captured whole-class conversation. A video-recorder was placed at the back corner of the room and ran continuously. The display from the overhead projector was in the field of view. The display from the view-screen, which was used in conjunction with the projector, was clearly visible on the video-recording.

In starting to write this paper, I scanned the transcripts of the audio-recordings for mention of practices that were 'quick, quicker, quickest, saves time, easy, easier, and easiest' and limited the analysis to practices so described. The fieldnotes and video-data provided details of calculator screen-displays.

The unit of analysis in the paper is the class. I discuss practices in the class, but this does not imply that all students adopted them. So as not to be misleading, I report the frequency with which the practices were evidenced in the various data. Because I could view the actions of only one or two students at any one time, the figures understate actual implementation of calculator strategies in the class. Another limitation of the paper is that, because the observation period extended over one fifth of the calculus course, a small sample only of approaches to calculation used by the students is discussed.

4. THEORETICAL REFERENTS

Literature on operational and structural understanding, algebraic understanding, and technical understanding needed for effective calculator use informed analysis. A short review follows.

4.1 Operational and structural understanding

All types of representation can call forth operational and structural understanding (Sfard, 1991), which are also known as procedural and conceptual understanding (Hiebert and Carpenter, 1992). Thus, numerical, algebraic and graphical forms can be interpreted in terms of the operations that underpin them, or as entities, which have properties and can be operated on. For example, $f(x) = 3x$ can be interpreted as implying a multiplication operation, which produces function values; or might be recognised as representing a linear function, which can be differentiated.

Success in mathematics depends on the ability to flexibly interpret all representations in terms of operations and as structures (Gray and Tall, 1994; Sfard, 1991). Moreover, representations may not be neutral in regards to the interpretations that they invite. In particular, formulae saved on technology, without any calculation processes showing, demand interpretation as structures. Schneider and Peschek (2002) give the example of the formula ew(r, i, n) for an annuity. The properties of the structure named 'an annuity' can be explored by changing the values of r, i, and n and examining the output values of ew(r, i, n). Function graphs presented in their entirety are likely to elicit structural interpretations about their shape and main features (Sfard, 1991), while graphs produced dynamically on technology are more likely to elicit operational interpretations about the processes of their construction (Forster, 2004). In summary, I distinguish the operational understanding and structural understanding that were called for in the Year12 students' methods of calculation.

4.2 Algebraic understanding

Sfard and Linchevski (1994) describe how algebraic thinking began, historically, with finding single, unknown values in problem solving, and reasoning was expressed in words and numerals. Letters appeared in written computation circa 250 A.D. and, in line with the early thinking, each letter stood for an unknown, fixed value. Some sixteen centuries later, letters were introduced also for numerical givens in algebraic expressions and, subsequently, the idea evolved that letters could stand for changing magnitudes, which gave rise to functional algebra. The terms 'generalised number' and 'variable number' were proposed for the changing components of algebraic expressions but, according to Sfard and Linchevski (1994), they were discarded as imprecise. Rather the term 'function' was used for the expressions in their totality.

The problematic term 'variable' is used widely now in relation to letters in algebraic expressions and has attracted a variety of definitions. For instance, in a major study into students' algebraic understanding, Kuchemann (1981) said that students viewed letters as variables if they treated them as standing for a range of unspecified values. Kuchemann also observed that students evaluated letters (i.e., assigned numbers to them) when solving algebraic equations, then, checked if the numbers worked, thus, they treated the letters as knowns. Otherwise, they: treated letters as objects, which did not necessarily have numerical significance (e.g., 'a' for apple); viewed that letters stood for specific (unique) unknowns, and viewed that they stood for generalised numbers having several values. Treatment as objects, without any recognition of numerical value leads to anomalous interpretations, for example, that $a + b$ cannot equal $b + c$.

Sfard and Linchevski's (1994) account provided me with a basis for thinking about the conceptual demands of different algebraic interpretations but, in the analysis, I intend Kuchemann's (1981) wider range of definitions when I refer to letters. I use the term parameter for numerical givens in algebraic expressions, yet recognise that parameters can also be afforded generalised number and variable definitions (Drijvers and Haarwarden, 2000). However, the higher-level uses were not relevant to the examples of calculation that I give.

4.3 Algebraic and technical understanding in relation to calculator use

Accounts by Schwarz and Dreyfus (1995), Pierce (2001), and Heck (2001) on algebraic understanding in relation to the use of technology, and Guin and Trouche's (1999) views on technical understanding also informed my analysis. Schwarz and Dreyfus (1995) define three categories of skill for dealing with algebraic and other information in computerised environments. The categories comprise skills for: dealing with partial data (e.g., recognising that points graphed as pixels belong to a continuum of data points); linking information in numeric, algebraic and graphical settings; and transforming numeric, graphical and algebraic information without changing the type of information (e.g., operating on functions to create new functions).

Pierce (2001) defines five types of algebraic understanding that are essential for effective calculator use. These are: recognition of conventions and basic properties (e.g., the meaning of symbols and order of operations); identification of structure (e.g., factors); identification of key features (e.g., dominant term); linking symbolic and graphical representations; and linking symbolic and numeric representations.

Heck (2001) describes the use of 'variables' on computer technologies. He uses the term loosely so that it encompasses specific value, generalised number and variable interpretations. He distinguishes that computer algebra systems allow numbers, function expressions, equations, inequalities, lists and tables of numbers, computed algebraic results to be stored and recalled by using the appropriate 'variables' (or memory names). The HP38G graphics calculators that the Year 12 students used afforded some of these possibilities. In particular, students utilised stored functions by naming their location F1, F2 etc, which I refer to as function identifiers because the term variable was not always appropriate.

Guin and Trouche (1999), amongst others, define domains of technical knowledge for effective use of technology. Such knowledge includes knowing the capabilities that are available and how to access them; knowing syntax; and dealing with anomalous graphical displays. Artigue (1996, 2002) notes that appropriating technology for mathematics takes time and depends in large part on explicit identification of calculation capabilities in class; and that patterns of use may change over time as expertise develops. Furthermore, impasses, errors and inefficiencies may be due to inadequate technical knowledge (Guin and Trouche, 1999) and/or to limited mathematical understanding, particularly of variable (Drijvers and Herwaarden, 2000; White and Mitchelmore; 1996) The various domains of algebra understanding and technical expertise discussed above were all relevant to the Year 12 students' calculation methods.

5. ANALYSIS

5.1 Transporting results and expressions

In the Year 12 class, one means to quick computation was transporting results of calculation into subsequent calculations. Students achieved this in four ways. Examples are given below. Discussion follows on benefits of the methods and the algebraic understandings that underpinned them.

The methods

The simplest method was proceeding with calculation when interim results were obtained. See Figure 1 which shows a student's solution for k in $5 = 4e^{1.03k}$. After producing the logarithm result, she keyed '/'. This activated 'Ans' , which carried down the .2231... result.

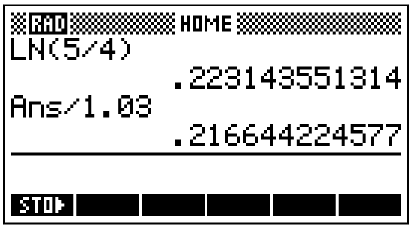

Figure 1. Transporting values using 'Ans'

Second, students highlighted results and used the COPY key. See Figure 2 which relates to an exercise on exponential growth. It shows a student's calculation for the amount to which $100 accrued after one year with an interest rate of 100% per annum and by-the-minute compounding. The alternative that some students used of entering the expression for the amount, 100*(1+1/(365*24*60))^(365*24*60), took more key-strokes to enter and required brackets around both occurrences of 365*24*60.

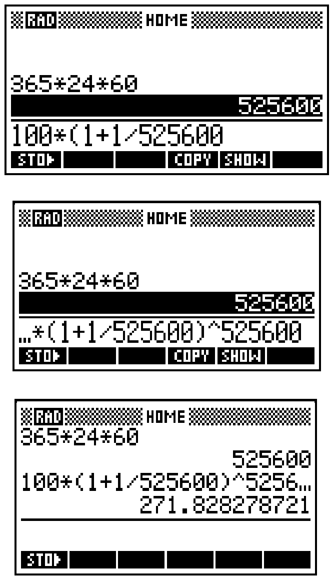

Figure 2. Transporting values by using the COPY capability.

Students also used the highlight and COPY procedure for numeric and algebraic expressions, then, edited the expressions for use in ongoing calculation. Figure 3 shows a student's evaluation of $\int_0^3 (-5p/2)\cos(p\ t/2)\mathbf{i} + (-5p/2)\sin(p\ t/2)\mathbf{j}\,dt$.

She calculated $\int_0^3 (-5p/2)\cos(p\ t/2)dt$, then, copied it and changed cos to sin in order to evaluate $\int_0^3 (-5p/2)\sin(p\ t/2)dt$.

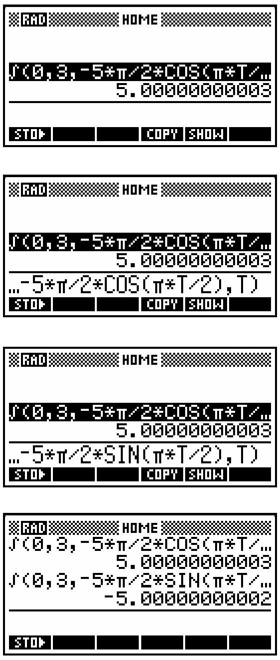

Figure 3. Transporting an expression by using COPY and then editing the cos operation.

The third method was naming the memories in which the results were stored. See Figure 4 which shows a student's solution for X at t = 30, given $|X-25| = 75e^{-kt}$ and X = 60 at t = 15. She used the generalised equation $A = \mathrm{I}\,\mathrm{e}^{kt}$ that she had stored in the solve aplet of her calculator (first screen, Figure 4). She treated A as standing for $|X-25|$, solved for k (second screen), solved for A at t = 30 (third screen), and then changed to the home screen in order to deal with the absolute value (fourth screen). Other stored values that I saw students use were ROOT for a root that had been located with the automated facility on function graph, ISECT for the x co-ordinate of the point of intersection on a graph, and T for the t value on which the cursor had most recently been placed on a parametric graph.

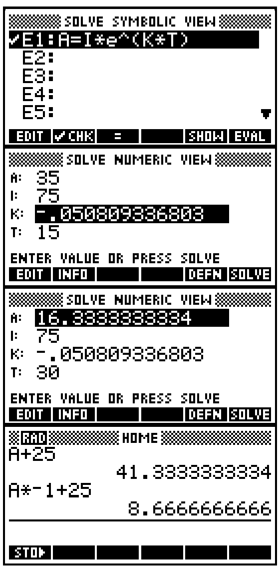

Figure 4. Calculation using the value in memory 'A'.

The fourth method of transportation was naming the location of functions using the identifiers F1, F2 etc, and specifying the value of the variable. See Figure 5 which illustrates a student's calculation for speed on $\mathbf{r}(t) = 10\sin t\,\mathbf{i} + 5\sin 2t\,\mathbf{j}$ at $t = p/2$. The components of velocity were stored in her calculator from a previous graphing activity. ABS was the command for absolute value.

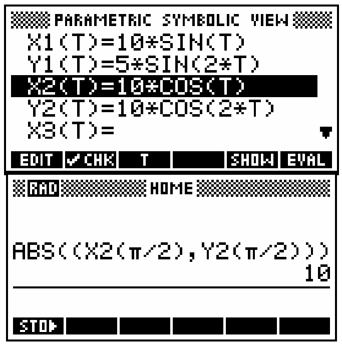

Figure 5. Calculating speed by naming the location of the velocity components.

Otherwise students used function identifiers with the appropriate variable (e.g., F1(X), X1(T)) to refer function expressions. See Figure 6 where he task was to determine the area bounded by $f(x) = \cos x/(2+\sin x)$ and the x-axis. The student had entered $f(x) = \cos x/(2+\sin x)$ in the graphing facility of her calculator to see where the area lay. Then, she calculated the area in the home screen using F1(X).

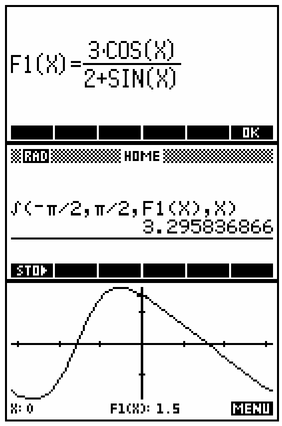

Figure 6. Importing a function into the home screen.

Quickness and other benefits

The benefits of transporting numerical values using the Ans and COPY capabilities (Figures 1-2) were: transcribing was avoided, all the decimal places were carried through, and less key-strokes were needed. In addition, because the Ans and COPY capabilities were available, it was practical and efficient to break up and think through calculation in parts; the parts could be reviewed (see Figures 1 and 2); and there was less need to use brackets to preserve order of operations on the calculator (see Figure 2), which can be a source of error. These advantages, however, depended on knowledge of order of operations to break up calculation appropriately.

One disadvantage of the Ans functionality was that it caused error when activated unwittingly. For instance, if values were highlighted in the solve aplet (Figure 3) and multiplied/divided etc, the value operated on was Ans, the last value in the home screen. Thus, unproductive contingencies as well as affordances of the functionalities had to be learnt.

The benefit of copying and editing numerical and algebraic expressions (Figure 3) was saving key-strokes. The procedure was useful when expressions differed only slightly, as for sine and cosine. In vector work, ABS was easily changed to ARG, for the argument or angle in standard position.

While Ans and COPY were used *within* the home screen, the methods in Figures 4 and 5 affected the transportation of values *into* the home screen from the solve and parametric aplets respectively. Imported is a better descriptor because movement was between different parts of the calculator. Rounding, transcribing and key-stroke benefits applied.

The method utilising F1(X) notation (Figure 6) had the effect of importing a function expression, or students may have held an image of the function graph when using the notation. In either case, re-entry of the expression was avoided. However, the HP38G calculator took longer to perform the calculation with complicated functions than if expressions were re-entered, so students needed to weigh up the advantage and disadvantage.

Students sometimes moved to COPY and EDIT for evaluation of expressions using different values of a variable. Copying was quicker than re-keying the expression, but the teacher discouraged it. Instead, he favoured using tables on the calculator if several function values were needed. This required formulation of functions to produce the table. For example, when a student suggested copying in relation to Figure 5 to determine speed at different times, the teacher asked for a quicker method. Consequently, functions for calculating speed, argument in radians, and argument in degrees were developed through class discussion (see Figure 7). Speed and angle values at different times were obtained from the table, and the speed graph was discussed. At other times the teacher encouraged use of calculator graphs or the solve aplet or for repeated calculation (e.g., see Forster et al., 2002).

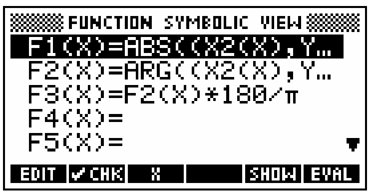

Figure 7. Speed F1(X), argument in radians F2(X) and argument in degrees F3(X). The calculator was set in radians. F1(X)=ABS((X2(X),Y2(X))) and F2(X)=ARG((X2(X),Y2(X)))

The application of function identifiers shown in Figure 7 was not isolated. A student showed me two other set-ups that she had stored in her calculator and had been developed in class. One was for producing tables and graphs for first and second derivatives (see Figure 8). The other was for calculating the Newton Raphson method.

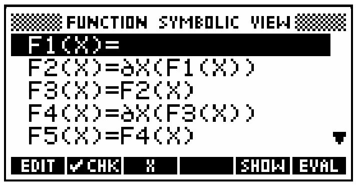

Figure 8. Use of function identifiers for differentiation

In summary, quickness in calculation in the Year 12 class was achieved through transporting numerical values, numerical expressions, and algebraic expressions within screens; and importing values and algebraic expressions between screens. The methods were not accepted uncritically—the affordances and contingencies were subjects of discussion in class and direction by the teacher.

Algebraic understanding

Algebraic understanding that underpinned the different methods ranged from simple interpretation of letters and words (such as Ans and ROOT), to interpretation of function relationships. For instance, in the solution relating to Figure 5:

(a) '*A*' and |*X* - 25| were used interchangeably;
(b) '*I*' was assigned a *known value*;
(c) '*K*' stood for a *specific unknown*, which was determined as a first step in the calculation;
(d) '*I*' and '*K*' were *parameters* whose values stayed fixed once they were determined;
(e) in the Solve aplet, '*A*' and '*T*' were treated as *variables* that were linked by the exponential relationship. They were treated as standing in a one-to-one relationship--for each value of '*A*' there was only one value of '*T*';
(f) '*A*' was treated in the Home screen as standing for a *known value* ($41.\dot{3}$).
(g) the student recognised that there was more than one solution for '*X*', in other words, she treated '*X*' as a *generalised number*;
(h) in performing the last stage of her solution, the student discarded the second value for X ($8\cdot\dot{6}$, see Figure 5) because it did not fit the practical context to which it related.

Thus, flexible interpretation of letters was called for in the single example-at the initial stage of entering data, in the middle stages of calculation and in interpreting the results. Absence of flexibility led to error. For instance, sometimes students failed to search for multiple solutions with polynomial and other equations when using the solve aplet. They assumed letters stood for unique values rather multiple values. Also, they accepted solutions without checking their validity: they omitted to substitute the numerical solutions into the algebraic expression or use the check that was available on the calculator, with the result that they sometimes gave answers for equations that had no solution.

The employment of function identifiers signified abstract reasoning in that function expressions were not on display. The notation stood for known numbers (e.g., X2(π/2)) and variables (e.g., F1(X)). The functions in variable form (F1(X) etc) were treated as objects and were operated on. One complexity which attracted comment was that the independent variable T in the parametric notation X2(T), Y2(T) had to be changed to X in the function aplet (Figure 8). Replacement of T with X was necessary for all rectilinear and vector motion calculations done in the function aplet.

Schwarz and Dreyfus's (1995) third skill category, transformation of representatives, was relevant to the use of letters and function identifiers (Figures 4-8). Operations were performed on algebraic representatives. Skills in Schwarz and Dreyfus' second skill category were also brought to bear: links were made between algebraic and numeric settings (Figure 5), and between algebraic/graphical/ numeric settings (Figure 6). These skills and the thinking associated with them are important for more advanced mathematics and for utilising the symbolic capabilities of graphics and CAS calculators. Execution of the skills on a graphics calculator could be valuable preparation for more advanced work.

Returning now to Figure 3, the task was integrating $(-5\pi/2)\cos(\pi t/2)\mathbf{i}+(-5\pi/2)\sin(\pi t/2)\mathbf{j}$. The component parts were treated as mathematical objects and operated on. Also, when the first component was copied and edited, it was treated as a material object without mathematical consequence. In fact, the student's integration method extended the capabilities of the HP38G calculator, which otherwise would not integrate vector expressions. The teacher recommended the approach when the student suggested it.

In summary, pursuit of quickness in calculation on the calculator depended on basic and more sophisticated algebraic understandings, including the treatment of functions as mathematical objects. Students potentially also consolidated and advanced their algebraic understanding, and ability to utilise and link different representations on the calculator, as they took up the 'quick' approaches and extended them. About the uptake, my fieldnotes and other data evidenced that: 'Ans' was used frequently and routinely. I noted that 9 of the 13 students use the COPY function, 5 imported values into the home screen using letters and words, and 6 of the 13 students imported values or expressions using the function identifiers. Predictably, the simplest action in terms of reasoning was most prevalent.

5.2 Selecting alternative approaches

The teacher and students often compared different solution paths on the basis of quickness and ease, for instance, repeated calculation in the home screen versus the table (see above). Quickness was also a criterion in selecting calculator commands, determining roots and maxima and minima, setting up the calculator for graphing, and choosing to use/not use calculator graphs. These facets of calculator use are discussed below.

Tables of values

During the time that I attended the class, tables on the calculator were accessed several times for the calculation of function values (as in Figure 7), where the purpose was investigating vector relationships early in the topic; and, on one occasion, the teacher asked the class to infer relationships between the **i** and **j** components of an **r**(t) function as displayed on the calculator (see Forster, 2003). Thus two benefits of calculator tables that are commented on in the literature were realised in the class; namely, tables were a site for active calculation (Schwarz and Dreyfus, 1995) and for discerning changes in function values (Schneider and Peschek, 2002). However, I recorded only one instance of a student spontaneously accessing a table to answer textbook questions, which mainly involved the application of vectors relationships to solve simple problems. Thus, it seemed that tables were used in the deduction of vector properties but not in the application of them. The infrequent use of tables by students reflects the findings of Pierce and Stacey (2001).

Calculator commands

The evaluation of definite integrals on the calculator was widespread and was recommended by the teacher. The ABS and ARG commands were also widely used. The syntax for them, for instance for 3**i** - 7**j**, was ABS((3,−7)) and ARG((3,−7)). The number of keystrokes was about the same as for the equivalent square root and arc tangent expressions, $\sqrt{\left(3^2+(-7)^2\right)}$ and ATAN(−7/3). Therefore, relative quickness of the alternatives was not based on key-stroke time. Rather, formulation was simpler because the square root and other operations did not have to be shown, and ARG always gave angles in the correct quadrant whereas ATAN did not.

ABS and ARG expressions invited structural interpretations (as objects) because no operations were visible, and this property most likely was influential in one student's innovative use of ARG. She moved spontaneously to calculate the angle between velocity and acceleration vectors by subtracting the argument for velocity from the argument for acceleration, in preference to using the conventional dot product formula $\cos\theta = \mathbf{a}.\mathbf{b}/|\mathbf{a}||\mathbf{b}|$. ARG stands for angle, so the method was more direct. The teacher recommended it to the class. The student also moved to calculate time for maximum height on a trajectory by graphing the argument of velocity as a function of time and finding the root, thus finding where velocity was horizontal. (Figure 9). This method was closely related to work discussed in class (Figure 8).

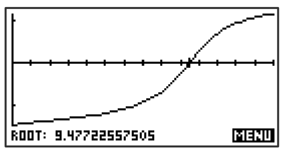

Figure 9. Graph of the argument of $\mathbf{v}(t)=(2t+1)\mathbf{i}+(t^2-8t-14)\mathbf{j}$

Hence, the ABS and ARG commands facilitated computation of magnitude and direction by all students. Both commands offered new possibilities for conceptualising vector relationships, and one student realised two possibilities for ARG. In fact, the possibility of operating on calculator commands opens up a plethora of possibilities for new methods and thinking in mathematics. Discussion in class on problem solving with the commands could be beneficial. Other innovative applications of ARG and ABS in the study of vectors, as evidenced in a Year 11 class, are reported in Forster (2000) and Forster and Taylor (2000).

Determination of roots and maxima and minima

Methods for determining roots and relative maxima and minima of functions received attention several times during whole class discussion, while I attended the class. For example, when the teacher asked the class how to find the time of impact on the ground for $\mathbf{r}(t) = 2t\mathbf{i} + (10 + 12t - 4.9t^2)\mathbf{j}$, the methods that students suggested were to isolate the $\mathbf{j}$ component and use: the POLYROOT command (Figure 10), the function graph with the ROOT command (Figure 11), and the solve aplet (Figure 12). A student argued for POLYROOT, and the teacher agreed: "It's quicker". The teacher also asked the class what problems could occur with the solve aplet and a student identified that only one answer is produced at a time. Thus, the common source of error was addressed.

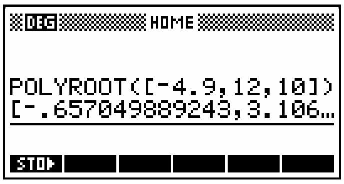

Figure 10. Solving $f(t) = 10 + 12t - 4.9t^2$ using POLYROOT

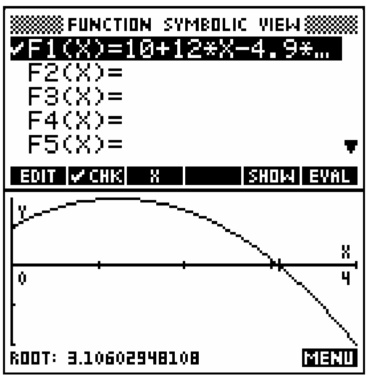

Figure 11. Solving $f(t) = 10 + 12t - 4.9t^2$ using the graph and ROOT command.

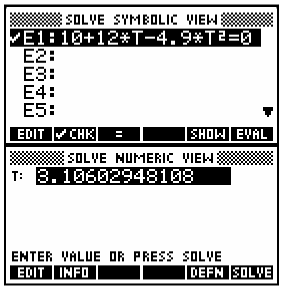

Figure 12. Solving $f(t) = 10 + 12t - 4.9t^2$ in the solve aplet.

The three methods for determining roots were all potentially faster than using the quadratic equation, which the teacher jokingly suggested using on another occasion. In general, using POLYROOT was a quick method for finding the roots of any polynomial equation. No operations were shown so the command was quick to formulate. An advantage over the other methods was that all roots were displayed. Students did not need to rely on algebraic/numeric reasoning about the number of roots and approximate values of them in order to choose values to start the iterative solutions in the solve aplet or to select scales to display the roots on the graph.

However, advantage depended on students knowing (a) the syntax, (b) the menu from which to retrieve the command so that they did not need to key POLYROOT, and (c) how to interpret the output, which included complex roots, depending on the function. The class had studied a unit on complex numbers and meeting these in the POLYROOT outputs did not seem to cause students difficulty. Rather, I noticed two students utilise the command to show that functions did not intersect the horizontal axis—thus relying on the property that the roots were complex.

Hence, relative quickness of the alternative methods for finding roots depended on students' algebraic and technical knowledge, and benefits other than quickness may have influenced their choices. The graph, for instance supported visualisation of the trajectory in the above example (see Figure 11). I noticed eight of the 13 students use POLYROOT, three graph the function, and three use the solve aplet. Some students used more than one of the methods and their choices are included in the counts. The limited sampling indicates that the most common method was POLYROOT. Unlike graphical solutions on technology, commands such as POLYROOT receive little attention in descriptions of graphics calculator use and may deserve more promotion in upper-secondary classes because the presence of complex roots is highlighted. Another method for finding roots is by-hand factorisation, which Pierce and Stacey (2001) observed students preferred for simple expressions.

Discussion on maxima and minima took a different turn to that for roots. For instance, when the teacher asked for methods to determine the maximum height for $\mathbf{r}(t) = 2t\mathbf{i} + (10 + 12t - 4.9t^2)\mathbf{j}$, a student said to plot the $\mathbf{j}$ component. The teacher recommended differentiating the $\mathbf{j}$ component was quicker. Students argued it was easier to plot the graph, there was more chance of error with the manual method, and that you still had to evaluate the maximum height, which made the manual method longer. So, the choice was between analytic and graphical approaches.

In the context of individual problem-solving, I noticed four students use a graph and three used differentiation when locating maxima or minima, and there was the instance when the student graphed the argument of velocity and used the ROOT facility to determine time for maximum height on a $\mathbf{r}(t)$ trajectory (Figure 9). Thus, I did not observe a distinct preference in the methods chosen for determining maxima and minima in the class, and the teacher and students did not agree on which was more efficient. In my view, neither was superior. Ability to differentiate the relatively simple functions in the questions and understanding that the derivative was zero at the turning point were not in question.

However, adopting calculator methods can mediate against understanding in a subject domain. A case in point is limits (Lagrange, 1999). Second, availability of a representation does not guarantee sensible use of it (Schneider and Peschek, 2002). Instances were given earlier in regards to the solve aplet. Third, teachers make choices and specify pen and paper techniques but may leave students to decide among the multiple calculator techniques themselves, which can be counterproductive (Artigue, 2002). Multiple methods, however, may broaden understanding in a domain, and different methods may suit different problems. All the issues warrant consideration when calculator methods are introduced.

Setting up the calculator for graphing

When graphing with the calculator, the teacher was dogmatic about choosing scales before plotting. For function $f(x)$ graphs, this meant choosing the range of x values over which to graph, which relied on knowledge of the structure of the graphs and mental calculation, for example estimation of roots. Then, the 'Autoscale' command was activated, which produced graphs showing the maximum and minimum function values in the chosen domain (see Figure 6). Thus, graphing, if efficient, started with algebraic thinking.

A more complex routine was used for $\mathbf{r}(t)$ functions and it was decided by the class, during the time that I spent with the class. The teacher did not hand down the strategies but rather he led the class in determining them when problems arose with graphing. The strategies addressed limitations of the digitised plotting (see Forster, 2003).

Evidence of adherence to the strategies was that seven out of nine students whom I noticed starting to graph chose scales appropriately and two did not. The two pressed PLOT without accessing the plot set-up. Selecting scales before plotting did not guarantee adequate graphs were obtained. Algebraic thinking was needed in judging adequacy and making adjustments.

Visualisation with graphs

Multiple claims were made that graphical approaches on the calculator were easier than analytic approaches, or that visualisation with calculator graphs made getting started on analytic solutions easier. Students said graphical approaches were easier than other methods for solving several parts of a rectilinear motion question and converting a quadratic function to turning point form. The teacher said area problems were easier if graphs were used to determine the domains over which to integrate and, on four occasions, he said that analysing movement on circuits defined by $\mathbf{r}(t)$ functions was easier if graphs were used.

Students' written solutions indicated that they used graphs frequently in area and volume problems. The proportions that were hand-generated and calculator-generated were not clear. In regards to the teacher's recommendations for visualisation with $\mathbf{r}(t)$ graphs on the calculator, I observed only four students use a calculator graph to assist a solution during individual problem-solving, in the 12 lessons on the vector topic. An explanation for the low utilisation of $\mathbf{r}(t)$ graphs on the calculator is that students did not need them to help determine processes of solution because early work in the topic had established properties on which to draw. For instance, I noticed only one student plot the $\mathbf{r}(t)$ graph to get started in determining when a particle on $\mathbf{r}(t) = (t^2 - 5t + 1)\mathbf{i} + (1 - 14t + t^2)\mathbf{j}$ travelled vertically and horizontally.

Section summary

Tables and graphs on the calculator were recommended in the class as means for achieving quick and easy solutions. In practice, tables and graphs were used in the vector topic for investigating properties, but it seemed that students used them minimally in vectors problems. They pursued analytic methods and calculation in the home screen instead. This is consistent with Artigue's (1996) finding that, if students use CAS efficiently, they favour symbolic capabilities and reserve graphing for testing conjecture. However, the Year 12 students utilised calculator graphs selectively, including for the determination of roots and maxima and minima. One aspect of efficient graphing was choosing the scales, which relied on algebraic/numeric reasoning.

The ABS, ARG and POLYROOT commands were means to quick calculation. Use of them meant that the Year 12 students did and thought about mathematics in non-traditional ways. Formulation was relatively simple because operations were not included. The absence of operations meant the commands were conducive to being treated as objects in calculation and this possibility was taken up with ARG.

5.3 Social promotion of methods

Class discussion on the methods described above made them available to the class. The teacher provoked discussion by asking for methods rather than specifying them and by sometimes asking for quicker alternatives when students made suggestions. He made claims about quickness which students sometimes challenged. Episodes of discussion and debate in the class are provided in Forster *et al*. (2002) and Davis and Forster (2003). They highlight students' sophistication in mathematical debate and, importantly, ability to debate develops through debating (Young, 1992). Hence, the pursuit and critical attitudes to quickness potentially fostered the communication skill. The teacher also brought individual student's innovative /efficient solution approaches to the attention of the class.

Use of the view-screen was a feature of whole-class work. Students' calculators were connected in 10 of the 21 lessons that I observed, and the teacher's calculator in three lessons. The lead on the panel was unusually long, so that students stayed at their desks when their calculator was connected and, so, could still take notes and discuss work with peers. Another outcome of the set-up was that students at the back as well as the front had their calculators connected—eight different students had turns while I observed the class. The practice was relatively inclusive.

Students used the view-screen to display approaches that they had decided individually, and they performed calculation to support the teacher's instruction. Sometimes the teacher asked students to connect their calculators while solving set tasks as part of whole-class work. Others in the class could check their working against the display, use it to get started, and ask questions about the display.

The display seemed to provoke participation in whole-class conversation. As well, it meant that students did not have to rely on spoken descriptions of calculator methods that were ambiguous and unclear. Syntax problems, anomalies and inefficiencies emerged as students performed calculation, and the teacher led discussion on the calculator problems when they appeared.

Often the student whose calculator was attached to the panel led conversation by forging ahead with calculation, in which case the teacher typically gave a commentary on what she was doing. His commentary moderated the pace of calculation, for the benefit of the class. Other times students in the class made suggestions or the teacher gave directions, and the student operating the calculator responded. One problematic aspect was that occasionally a student would say the result of calculation before it appeared on the whiteboard, which disrupted proceedings.

When the teacher connected a student's calculator, often other students picked up their own calculators and twice while I was observing the class the teacher directed students towards this end. In this mode, students were active in calculation, which was potentially more valuable to them than watching and listening only. There were several instances when students compared their own displays with the projected one, found that they differed and voiced this, so that a variety of methods and understandings became available to the class. The practice of students using their own calculators while one student's calculator was attached to the view-screen seemed to me to be beneficial for individuals and the class.

Guin and Trouche's (1999) comments about the view-screen with a student's calculator attached are highly pertinent. Namely, that the student plays a central role as guide, assistant and mediator in the mathematics being discussed; the set-up favours classroom debates, allows the teacher to become aware of the student's expertise/lack of expertise with the calculator, and requires that the teacher has a thorough knowledge of the calculator. Also, the set-up allows the teacher and other students to control the activity of the student with the calculator (Trouche, 2000).

The action was different from Doerr and Zangor's (2000) and Goos, Galbraith, Renshaw and Geiger's (2000) descriptions of use of the view-screen, where students came to the front of the class and reported group-work and led discussion. The Year 12 students of my inquiry did not need to come to the front because of the long lead, and the teacher did not organise group-work, although students sat themselves in four groups and discussed their work.

In summary, conversation that was focussed on the projected display was the major means through which calculator methods were made public and interrogated for efficiency. The spoken and visual stimuli were complementary. Students' calculations rather than the teacher's were topics of discussion, and simultaneous use by students of their own calculators meant they participated in activity rather than being passive observers.

6. CONCLUSION

Critical attitudes to quickness in calculation characterised the practice in the Year 12 class. Quickness was achieved through transporting values and expressions within a screen, and importing values and expressions between screens. The number of key-strokes was reduced, transcribing was circumvented, and accuracy was maintained. The approaches are recommended on these counts.

Inquiry highlighted that quick calculation relied on conventional knowledge of arithmetic and algebra-when formulating calculation, at interim stages of calculation, and in interpreting the calculator outputs. The calculator performed procedures but algebraic knowledge remained essential. The most sophisticated methods involved operating on functions using the calculator function notation F1(X) etc., which facilitated generalisation and involved co-ordinating algebraic, numerical, and graphical representations. The observed heavy reliance on algebra reconfirms the finding of others that algebra skills need attention if students are to use graphics calculators effectively and efficiently.

Efficiency also relied on calculator-specific knowledge, in particular of: syntax; limitations of calculator functioning; how to access commands; how to select scales to produce adequate graphs; and the relative efficiency of alternative approaches. Productive alternatives included using: tables for repeated calculation, the POLYROOT command for finding roots, and the ABS and ARG functions for magnitude and angle calculations. However, efficiency depended on the purpose in calculation and the status of an individual's understanding. For instance, calculator graphs were used in the discernment of vector properties, but students did not seem to utilise the visual support that the graphs offered in application questions. Thus, efficient use of calculators is evolutionary and not fixed.

The POLYROOT, ABS and ARG commands were devoid of operations, which simplified formulation and meant that they invited operations being performed on them. There were instances of this occurring. In view of the findings, a recommendation is that discussing the treatment of commands as objects could lead to greater appreciation of mathematical structure and to greater innovation by students.

A feature of the Year 12 teacher's teaching was that he did not leave students to discover quick methods themselves, albeit students did propose new methods for the class. Use of the view-screen with a student's calculator, and simultaneous use by others of their calculators, meant the teacher and students together identified efficient methods and addressed inefficiencies. The arrangements seemed to favourably mediate debate, learning and performance. So, final recommendations are that a view-screen should be a priority when calculators are included, and the arrangements for its use in the Year12 class deserve consideration.

REFERENCES

Artigue, M. (1996). Computer environments and learning theories in mathematics education. *In Teaching Mathematics with Derive and the TI-92, Proceedings of the International Derive and TI-92 Conference*, pp 1-17. Zentrale Koordination Lehrerausbildung, Munster.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning,* **7**, 245-274.

Curriculum Council (2003). *Syllabus manual Year 11 and 12 accredited subjects*. Curriculum Council: Osborne Park, Western Australia.

Davis, C. and Forster, P. (2003). Classroom communication in the classroom, including the use of open-ended questions. *Australian Senior Mathematics Journal,* **17 No 2**, 36-47.

Doerr, H. M., and Zangor, R. (2000). Creating meaning for and with the graphing calculator. *Educational Studies in Mathematics,* **41 No 2**, 143-163.

Drivjers, P., and van Herwaarden, O. (2000). Instrumentation of ICT-tools: The case of algebra in a computer environment. *International Journal of Computer Algebra in Mathematics Education,* **7 No 4**, 255-275.

Forster, P. A. (2000). Process and object interpretations of vector magnitude mediated by use of the graphics calculator. *Mathematics Education Research Journal,* **12 No 3**, 176-192.

Forster, P. A. (2004). *Conceptualisation through dynamic graphing*. Manuscript submitted for publication.

Forster, P. A., and Mueller, U. (2001). Outcomes and implications of students' use of graphics calculators in the public examination of calculus. *International Journal of Mathematical Education in Science and Technology, 32*(1), 37-52.

Forster, P., and Mueller, U. (2002). Assessment in calculus in the presence of graphics calculators. *Mathematics Education Research Journal,* **14 No 1**, 16-36.

Forster, P. A. and Taylor, P. C. (2000). A multiple perspective analysis of learning in the presence of technology. *Educational Studies in Mathematics*, **42 No 1**, 35-59.

Forster, P. A. and Taylor, P. C. (2003). An investigation of communicative competence in an upper-secondary class where using graphics calculators was routine. *Educational Studies in Mathematics,* **52 No 1**, 57-77.

Forster, P., Taylor, P. and Davis, C. (2002). "One hundred and sixty geckos and one without a tail": Wit and student empowerment when class discussion is centred on the display from a student's graphics calculator. *Australian Senior Mathematics Journal,* **16 No 1**, 56-64.

Fraser, B.J. and Tobin, K. (1989). *Exemplary Practice in Science and Mathematics Education*. Key Centre for School Science and Mathematics, Curtin University of Technology, Perth, Western Australia.

Gage, J. (2002). Using the graphic calculator to form a learning environment for the early teaching of algebra. *The International Journal of Computer Algebra in Mathematics Education,* **9 No 1**, 3-27.

Goos, M., Galbraith, P., Renshaw, P. and Geiger, V. (2000). Reshaping teacher and student roles in technology-enriched classrooms. *Mathematics Education Research Journal,* **12 No 3**, 303-320.

Graham, A. T. and Thomas, M. O. J. (2000). A graphic calculator approach to understanding algebraic variables. *In Proceedings of TIME 2000, An International Conference on Technology in Mathematics Education*, pp 137-144. University of Auckland and Auckland University of Technology, Auckland.

Gray, E. M. and Tall, D. O. (1994). Duality, ambiguity, and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education,* **25**, 116-140.

Guin, D. and Trouche, L. (1999). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning,* **3**, 195-227.

Heck, A. (2001). Variables in computer algebra, mathematics and science. *The International Journal of Computer Algebra in Mathematics Education,* **8 No 3**, 195-210.

Hiebert, J. and Carpenter, T. P. (1992). Learning and teaching with understanding. In ed. Grouws D. A.. *Handbook of research in mathematics teaching and learning.* pp 65-97. Macmillan, New York.

Kuchemann, D. (1981). Algebra. In ed. Hart K. M.. *Children’s Understanding of Mathematics* pp 11-16. John Murray, London.

Lagrange, J-B. (1999). Complex calculators in the classroom: Theoretical and practical reflections on teaching precalculus. *International Journal of Computers for Mathematical Learning,* **4 No 1**, 51-81.

Pierce, R. (2001). Using CAS-calculators requires algebraic insight. *Australian Senior Mathematics Journal*, **15 No 2**, 59-63.

Pierce, R. and Stacey, K. (2001). Observations on students' responses to learning in a CAS environment. *Mathematics Education Research Journal*, **13 No 1**, 28-46.

Schneider, E. and Peschek, W. (2002). Computer algebra systems (CAS) and mathematical communication. *The International Journal of Computer Algebra in Mathematics Education*, **9 No 3**, 229-242.

Schwarz, B. B. and Dreyfus, T. (1995). New actions upon old objects: A new ontological perspective on functions. *Educational Studies in Mathematics*, **29**, 259-291.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, **22 No 1**, 1-36.

Sfard, A. and Linchevski, L. (1994). The gains and pitfalls of reification-the case of algebra. *Educational Studies in Mathematics*, **26**, 191-228.

Tall, D. O., and Thomas, M. (1991). Encouraging versatile thinking in algebra using the computer. *Educational Studies in Mathematics*, **22**, 125-147.

Trouche, L. (2000). New technological environments: New constraints, new opportunities for the teacher. *The International Journal of Computer Algebra in Mathematics Education*, **7 No 3**, 165-179.

White, P. and Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, **27 No 1**, 79-95.

Young, R. (1992). *Critical theory and classroom talk*. Multilingual Matters: Adelaide, South Australia

BIOGRAPICAL NOTES

Pat Forster has wide experience teaching mathematics in the secondary and post-secondary sectors and in 2001 competed her doctorate in mathematics education at Curtin University of Technology, Western Australia. She was awarded an Australian Research Council Discovery Grant in 2003 which funds her present employment as a Research Fellow at Edith Cowan University, Western Australia. Her main research interests are classroom interactions and the use of computer technologies in the mathematics classroom. Her present research focus is statistics teaching and learning with technology in upper-secondary classes.

Ideas for Teaching and Learning

Computer Algebra versus Manipulation

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In the UK there is increasing concern about the lack of skill in algebraic manipulation that is evident in students entering mathematics courses at university level. In this note we discuss how the computer can be used to ameliorate some of the problems. We take as an example the calculations needed in three dimensional vector analysis in polar coordinate systems, and argue that these form a deterrent to many students. We illustrate how a computer algebra package may be used to shorten the length and complexity of these calculations with only modest effort from the teacher.

1. INTRODUCTION

A huge amount of material has been written about the use of technology in the teaching of undergraduate mathematics. In previous papers (Crowe & Zand, 2000a, 2000b and 2001) we surveyed the state of the art at the turn of the millennium. The breadth of software is enormous, ranging from commercial computer algebra packages to custom-built multimedia aimed at specific topics. However the use of custom-built multimedia packages is rare, partly because of the high cost of production. Until recently the cost argument could also be used against the 'high-end 'computer algebra packages. However student versions are becoming affordable, and there is now increasing competition which should help to keep costs down. Of course money is not the only (or perhaps even the most important) cost here: there is also the question of the time that the teacher needs to invest in developing resource material. In this short note we hope to indicate how a modest investment of effort can bring significant rewards.

Our primary experience is with distance education, which presents additional challenges to the teacher. For example, the use of a computer algebra package in a distance learning course must be mediated by the fact that expert assistance is not readily available to the student, so that a trivial syntax error can suddenly assume major proportions. Nevertheless, the benefits generally outweigh the costs. In particular it is widely recognised that graphics offer enormous opportunities for students to visualise, and that complicated numerical computations need not be avoided. In the latter context, mathematical models can be made more realistic by the incorporation of details that would be beyond hand-calculation. There is a substantial literature dealing with the educational aspects of using such technology, and calculus is perhaps the most quoted scenario. In this note we adopt a rather different perspective, which nevertheless seems to resonate with colleagues in other institutions. This is based on the empirical fact that the manipulative ability of students entering mathematics courses has declined over the years, and is a potential obstacle in certain parts of the syllabus. This problem was clearly identified in the London Mathematical Society's report of 1995 'Tackling the mathematics problem' (London Mathematical Society, 1995), and is also referred to in the recently published Smith report into post-14 mathematics education in the UK (Smith, 2004) which mentions "pupils' poor facility with the basic processes and calculations of mathematics". The methods used to teach mathematics in schools have changed and so it is not surprising that the skills of emerging pupils are different. We pass no judgement on the merits or otherwise of the post-60s approach to teaching in school. We do however, have to recognise the real entry characteristics of our students and plan accordingly. This note discusses how computer algebra packages may be used to this end.

2. IMPACT ON THE CURRICULUM

The use of the computer to minimise tedious numerical calculations is well known - indeed it could be argued that numerical analysis became a main-stream subject of study precisely because this facility was available (first via mainframes, then calculators and finally on PCs). For example in Open University courses the subject is introduced conceptually using the Euler method, but attention turns rapidly towards the more effective fourth-order method of Runge-Kutta. However even with the Euler method hand calculations are kept to a minimum, with Mathcad being deployed to obtain 'real 'answers. Subsequently, the qualitative behaviour of solutions of non-linear differential equations is investigated, both numerically and graphically, in a way that would have been impossible 20 years ago. There are parts of the 'traditional 'syllabus that can also be profoundly affected. Kawski (2003) has recently reported on the use of Maple in courses on differential geometry – a subject that abounds in the manipulation of complex formulae. One of his best examples concerns curvature: the formula for curvature is well known to generations of mathematicians, but actually computing curvature in particular cases is virtually impossible without the help of a computer. Kawski describes the use of Maple to investigate curvature interactively; this also presents the interesting possibility of trying to predict the curve that has a given curvature function!

The influence of the computer on the syllabus is clear. However our interest here is in the influence it has on the skills that are taught. For example there has been considerable debate over the need for 'traditional 'skills associated with differentiation and integration. On the one hand there is little doubt that confidence in manipulative ability is an asset, and in itself can reinforce the learning process; Tall 's procept theory (Tall, 1997) may be seen in this light. On the other hand there is a powerful argument that since a machine can perform a much wider range of differentiations (and especially integrations)than any human, and is much less prone to making errors there is less need to acquire manipulative skill in these things. As an extreme example it has been argued that the rule for integration by parts should no longer be included in the curriculum at all. However physicists and applied mathematicians frequently use this in a theoretical rather than computational context, so we find this hard to accept.

The Open University makes serious efforts to minimise the dependence on manipulative skill. All students are supplied with standard tables of integrals and derivatives (these form part of a course 'handbook 'that the student may take into the examination). On a practical level throughout our entry level courses (and subsequent applied mathematics courses) Mathcad is used to minimise the need for a student to be good at manipulation. Interestingly this raises its own problems: some students come to the course expecting to meet manipulative challenges and are rather ill at ease when computer supplants pencil and paper. We report on this phenomenon elsewhere, but for the purposes of this paper we would like to demonstrate that one particular barrier can be removed by using the computer.

3. A SPECIFIC EXAMPLE

We shall examine one specific example, but in doing so we hope to show that a modest investment of effort in using the technology can bring great rewards. The problem we address is that of vector calculus in three dimensions. It is assumed that the student has some degree of facility with 3×3 matrices and trigonometric functions, together with the elementary rules for differentiation. We pick up the story after the student has been introduced to the idea of div, grad and curl in cartesian coordinates, and now wish to treat (for example) spherical polar coordinates. The calculations involved are routine but extremely tedious, and for students who lack confidence in manipulation this can be a decisive obstacle. Some never surmount this, and as a consequence a large area of applied mathematics remains forever closed to them. Yet this need not happen!

We shall show how a computer algebra package can be used to circumvent the daunting calculations while still retaining the underlying meaning. If one accepts Tall's model of procepts (Tall, 1997) (which we do) then it may appear that such a short-cut risks loss of student understanding, since the very process of computation leads to the encapsulation of the concept. Against this we would argue that, having previously studied vector calculus in the cartesian system, a student's necessary proceptual structure is already in place.

The package we use is Mathematica, but the same approach works with Maple, Mupad and Derive.
Assume then that the student has mastered the definition of

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

and that this can be written in matrix-like notation as

$$\nabla = \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix}^T \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}.$$

Now comes the problem of expressing grad in spherical polar coordinates (our choice of labels is shown in Figure 1),using the chain rule and associated Jacobian. (Of course all these, and more, are likely to be defined internally in the software package or one of its libraries. However the student will not understand their definitions at this point.)

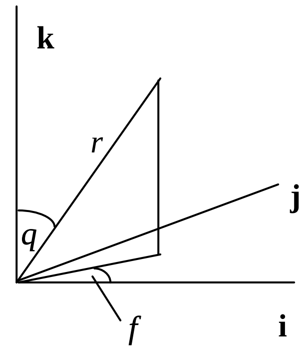

Figure 1. Polar Coordinate System

An unavoidable step is to express the polar unit vectors in terms of their cartesian counterparts - we shall retain the matrix notation.

$$\begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\partial \\ \mathbf{e}_f \end{bmatrix} = \begin{bmatrix} \sin q \cos f & \sin q \sin f & \cos q \\ \cos q \cos f & \cos q \ \sin f & -\sin q \\ -\sin f & \cos f & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix},$$

or $\mathbf{e}_{\text{pol}} = \mathbf{P}\mathbf{e}_{\text{cart}}$ say.

If orthogonal matrices have been taught previously then inversion is immediate. If not, then using the package is uncontroversial (Figure 2), and we have the expression $\mathbf{e}_{\text{cart}} = \mathbf{P}^{-1}\mathbf{e}_{\text{pol}}$ for **i, j** and **k** in terms of $\mathbf{e}_r$, $\mathbf{e}_q$, and $\mathbf{e}_f$.

$$\begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} \sin q \cos f & \cos q \cos f & -\sin q \\ \sin q \sin f & \cos q \ \sin f & \cos q \\ \cos f & -\sin f & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_q \\ \mathbf{e}_f \end{bmatrix},$$

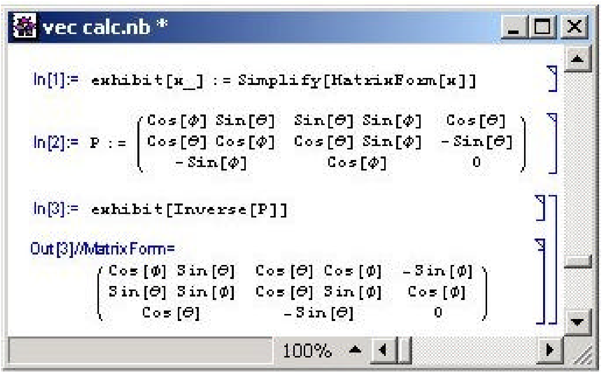

Figure 2. Inversion of spherical polar coordinate matrix

The next unavoidable step is to compute the Jacobian using the chain rule, which amounts to seeking expressions for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ in terms of $\frac{\partial}{\partial r}, \frac{\partial}{\partial q}$ and $\frac{\partial}{\partial f}$. In one direction (which is all we shall need) the computations are straightforward.

$$\frac{\partial f}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial f}{\partial z}$$

$$= \sin(q)\cos(f)\frac{\partial f}{\partial x} + \sin(q)\sin(f)\frac{\partial f}{\partial y} + \cos(q)\frac{\partial f}{\partial z},$$

$$\frac{\partial f}{\partial q} = \frac{\partial x}{\partial q}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial q}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial q}\frac{\partial f}{\partial z}$$

$$= r\cos(q)\cos(f)\frac{\partial f}{\partial x} + r\cos(q)\sin(f)\frac{\partial f}{\partial y} - r\sin(q)\frac{\partial f}{\partial z},$$

$$\frac{\partial f}{\partial q} = \frac{\partial x}{\partial q}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial q}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial q}\frac{\partial f}{\partial z}$$

$$= -r\sin(q)\sin(f)\frac{\partial f}{\partial x} + r\sin(q)\cos(f)\frac{\partial f}{\partial y}.$$

This leads to the equation

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial q} \\ \frac{\partial}{\partial f} \end{bmatrix} = \begin{bmatrix} \sin(q)\cos(f) & \sin(q)\sin(f) & \cos(q) \\ r\cos(q)\cos(f) & r\cos(q)\sin(f) & -r\sin(q) \\ -r\sin(q)\sin(f) & r\sin(q)\cos(f) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

or $\partial_{\text{pol}} = \mathbf{Q}\partial_{\text{cart}}$, say.

Once again (Figure 3) the package inverts this matrix with ease.

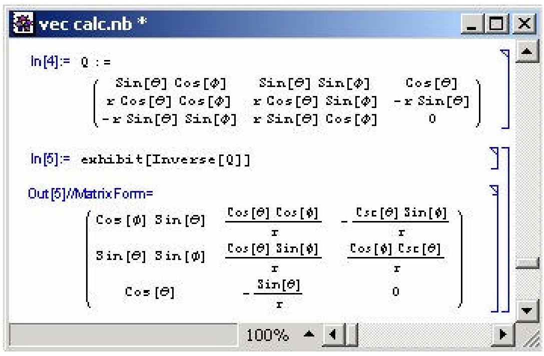

Figure 3. Inversion of the Jacobean matrix

and we obtain

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \sin(q)\cos(f) & \frac{1}{r}\cos(q)\cos(f) & \frac{-1}{r\sin(q)}\sin(f) \\ \sin(q)\sin(f) & \frac{1}{r}\cos(q)\sin(f) & \frac{1}{r\sin(q)}\cos(f) \\ \cos(q) & \frac{-1}{r}\sin(q) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial q} \\ \frac{\partial}{\partial f} \end{bmatrix}$$

or $\partial_{\text{cart}} = \mathbf{Q}^{-1}\partial_{\text{pol}}$. The correct formula for **grad** in spherical polar coordinates is now immediate

$$\nabla = \mathbf{e}^{T}_{\text{cart}}\partial_{\text{cart}}$$

$$= \left(\mathbf{P}^{-1}\mathbf{e}_{\text{pol}}\right)^{T}\left(\mathbf{Q}^{-1}\partial_{\text{pol}}\right)$$

$$= \mathbf{e}^{T}_{\text{pol}}\left(\mathbf{P}^{-1}\right)^{T}\mathbf{Q}^{-1}\partial_{\text{pol}}$$

and all that remains is a simple matrix computation (Figure 4) to arrive at

$$\mathbf{grad} = e_r\frac{\partial}{\partial r} + \frac{1}{r}e_q\frac{\partial}{\partial q} + \frac{1}{r\sin(q)}\frac{\partial}{\partial q}$$

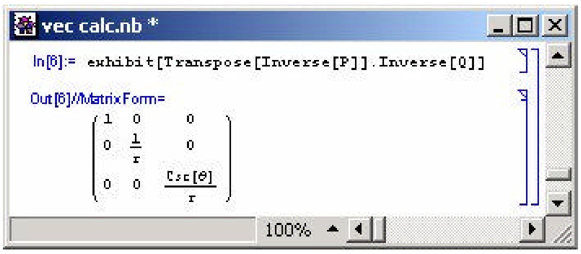

Figure 4. Computation of coefficient matrix for spherical polar form of **grad**.

With a little ingenuity, the computation of div, grad and curl can also be made transparent. For example suppose given a vector field in spherical polar coordinates

$$\mathbf{F} = \lfloor F_r, F_q, F_f \rfloor = F_r\mathbf{e}_r + F_q\mathbf{e}_q + F_f\mathbf{e}f$$

We must calculate

$$\nabla \times \mathbf{F} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r}\mathbf{e}_q \frac{\partial}{\partial q} + \frac{1}{r\sin(q)}\mathbf{e}_f \frac{\partial}{\partial q}\right) \times \left(F_r \mathbf{e}_r + F_q \mathbf{e}_q + F_f \mathbf{e}_f\right)$$

$$= \mathbf{e}_r \times \frac{\partial}{\partial r}\left(F_r \mathbf{e}_r + F_q \mathbf{e}_q + F_f \mathbf{e}_f\right) + \frac{1}{r}\mathbf{e}_q \times \frac{\partial}{\partial q}\left(F_r \mathbf{e}_r + F_q \mathbf{e}_q + F_f \mathbf{e}_f\right) + \frac{1}{r\sin(q)}\mathbf{e}_f \times \frac{\partial}{\partial f}\left(F_r \mathbf{e}_r + F_q \mathbf{e}_q + F_f \mathbf{e}_f\right)$$

Of course, in order to do this it is necessary to calculate the rates of change of the polar unit vectors in terms of the polar coordinates. Once again the package can do this with ease. (Figure 5). We recover the usual formulae.

$$\frac{\partial \mathbf{e}_r}{\partial r} = \frac{\partial \mathbf{e}_q}{\partial r} = \frac{\partial \mathbf{e}_f}{\partial f} = \mathbf{0}$$

$$\frac{\partial \mathbf{e}_r}{\partial q} = \mathbf{e}_q, \quad \frac{\partial \mathbf{e}_q}{\partial q} = -\mathbf{e}_r, \quad \frac{\partial \mathbf{e}_f}{\partial q} = \mathbf{0},$$

and

$$\frac{\partial \mathbf{e}_r}{\partial f} = \sin(q)\mathbf{e}_f, \quad \frac{\partial \mathbf{e}_q}{\partial f} = \cos(q)\mathbf{e}_f \quad \frac{\partial \mathbf{e}_f}{\partial f} = -\sin(q)\mathbf{e}_r - \cos(q)\mathbf{e}_q$$

Note the ease with which these results are obtained! The remaining computation, although rather unwieldy, amounts to several uses of the product rule, together with the information already to hand.

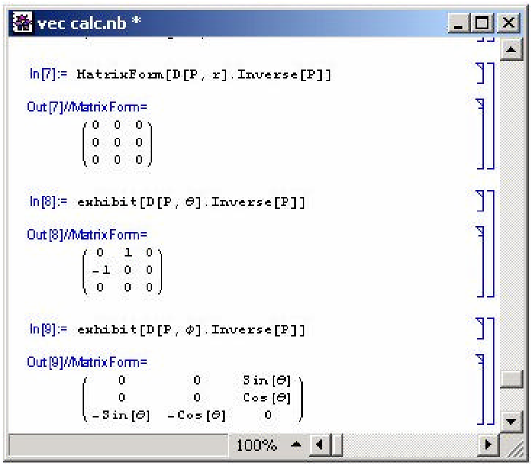

Figure 5. Computations of rate of change of polar unit vectors in terms of the polar coordinates

At this point the student is in a position to carry out vector analysis in polar coordinates, and has not been deterred by several pages of complicated calculations. We believe that students' confidence has been preserved, and they can move on to achieve familiarity via practice. The libraries built into the software can now be used, and the student understands where the results come from - a hurdle has been crossed.

4. CONCLUSION

In this note we have seen how a modest investment of effort in using technology can make some useful concepts and relationships more accessible to students of all abilities. It is clear that using a CAS should not be a mere add-on to the course; this would merely deter both students and the teachers. However, usages of the kind we have presented here recover a genuine learning opportunity which otherwise could been masked by tedious calculations. Indeed it would be useful to explore other areas of undergraduate mathematics to find similar examples.

REFERENCES

Crowe, D & Zand, H. (2000a) Computers and undergraduate mathematics 1: setting the scene. *Computers and Education* **35 No 2**: 95 –121

Crowe, D & Zand, H. (2000b). Computers and undergraduate mathematics 3: Internet resources. *Computers and Education*: **35 No 2** 123 –147

Crowe, D & Zand, H. (2001) Computers and undergraduate mathematics 2: on the desktop. *Computers and Education* **37 No 3**:317 –344

Kawski, M. (2003) Curvature for everyone. *Proceedings of the 9th ATCM conference*, Hsin-Chu

Smith, A. (2004) Making mathematics count. London, Department for Education and Skills

London Mathematical Society. (1995) Tackling the mathematics problem. London, London Mathematical Society.

Tall D. (1997). Metaphorical objects in advanced mathematical thinking. *Journal for Computers in Mathematics Learning* **2 No 1**:61 –65.

BIOGRAPHICAL NOTES

Hossein Zand had been a lecturer in the Institute of Educational Technology at the Open University for 25 years. He has an extensive knowledge of the use, and usability, of software in connection with undergraduate mathematics teaching. He combines this with other research interests, including ring theory.

David Crowe is a senior lecturer in mathematics at the OU (and has recently completed a term as sub-dean of the faculty). He is keen to advance the use of technology in the curriculum. Other research interests include theoretical computer science, particularly concurrency.

Ideas for Teaching and Learning

Playing with Powers

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This paper explores the wide range of pure mathematics that becomes accessible through the use of problems involving powers. In particular we stress the need to balance an applied and context based pedagogical and curricular approach to mathematics with the powerful pure mathematics beneath the simplicity of easily stated and understandable questions in pure mathematics. In doing so pupils realise the limitations of computational tools as well as gain an appreciation of the aesthetic beauty and power of mathematics in addition to its far-reaching applicability in the real world.

1 INTRODUCTION

In this paper we explore the curricular and pedagogical implications of the general area of "powers". By "powers" we mean the mathematics arising from the study of exponential functions. In the United States, the Algebra Standard recommends that students in grades 9-12 be introduced to the exponential function and classes of functions in general including polynomial, logarithmic and periodic functions (NCTM, 2000). Traditionally 9^{th} and 10^{th} graders (13-16 year olds) in the U.S are introduced to exponential functions by first extending the laws of exponents from integer powers to real powers. This is usually followed by graphing exponential functions such as 2^x, 3^x etc in order to convey the fact that for functions of the form a^x (a > 1), the domain is the set of real numbers whereas the range is $(0, \infty)$. The graph also reveals properties such as the increasing behaviour of the function and the *x*-axis being the horizontal asymptote for this function as $x \to -\infty$. Students are then introduced to the case where $0 < a < 1$ to study properties with which we are all familiar. A key topic in this traditional treatment of exponential functions is introducing students to the irrational number 'e' by studying the behaviour of $(1+ 1/n)^n$ as $n \to \infty$. Finally students are introduced to logarithms and the analogous laws of logarithms are derived from the laws of exponents.

In the mathematical experiences of the second author in Poland, as a high school student, exponential functions appeared slightly later in the curriculum in comparison to the U.S., at the beginning of the 3^{rd} year of secondary school (for students at the age of 16 to 17). The sequence of events was similar to the one described in the previous paragraph. First, students learned how to define powers with rational exponents so that all laws that hold for natural exponents were still satisfied. Then, looking at graphs, they were prompted to note that exponential functions map arithmetic progressions to geometric ones (and to use this property for graphing exponentials on graph paper) etc. Finally, powers with arbitrary real exponents were defined in a more or less precise way, using monotonicity. It is hard to say precisely how many students were able to digest and understand this section of the textbook. Not too many, we are afraid. In our opinion, the aim of the textbook was clearly, to purify and to "bourbakise" ... alas, with no real success. The Bourbaki were a group of mostly French mathematicians, who began meeting in the 1930s and aimed to write a thorough (formalised) and unified account of all mathematics. The "bourbakised" definition of $2^{\sqrt{2}}$ as the supremum of a suitable set of rational powers of 2 is not really entertaining, nor enlightening! Even mathematically oriented 16 years olds have lots of more interesting things to do! Pure mathematics should be, among other things, a source of fun – we shall come back to this later.

The Bourbaki essentially aimed to write a body of work based on a rigorous and formal foundation, which could be used by mathematicians in the future. For more information see Bourbaki, N. (1970). *Théorie des Ensembles de la collection elements de Mathématique*, Hermann, Paris or the Bourbaki website located at http://www.bourbaki.ens.fr/

In the traditional curriculum students also encounter word problems associated with exponential functions. Functions obtained by running an exponential regression on data arising from the natural, social and financial sciences are given *a priori* to the students in the context of solving these word problems. Numerous examples of such functions abound. For instance $p = 760e^{-0.145h}$ relates the pressure *p* on a plane (in millimetres of mercury) to height h (in kilometres) above sea level. The spread of information via mass media (TV and magazines) is modelled by $d = P(1-e^{-kt})$ where *P* stands for a fixed population and t denotes time (Coleman, 1964). Classical examples are of course found in physics. For instance Newton's law of cooling is given by $f(t) = T + (f_0 - T)e^{kt}$, where $k < 0$, f_0 is the initial temperature of the object that is heated, *T* is the temperature of the surrounding medium. Other classical examples are radioactive decay, growth of bacteria etc.

This approach has been reversed by reform-based curricula in the U.S, which emphasise modelling activities as a means of data generation. For instance bouncing balls with a known co-efficient of elasticity can be used. Students are asked to drop a ball (basketball, tennis ball etc) from a starting height and asked to keep track of the bounce height as the bounces progress until the ball is flat on the ground. Such data are interpreted by realising that the plot of bounce height over bounce numbers shows an exponentially decaying pattern and therefore results in the invoking of an exponential regression. For instance the data for a bouncing basketball is modelled fairly accurately by $f(x) = A\,(0.5)^x$ where A is the initial height and x is the bounce number. These are specific examples of the wider range of model eliciting activities (Lesh & Doerr, 2003) that expose students to applied mathematics in a real world context very early in their schooling experiences. This is in stark contrast to the "good old" days when one had to take a course in differential equations to encounter such modelling problems in a "recipe driven" didactic environment.

Exposure to the applied aspects of mathematics conveys to students only one aspect of the spectrum of mathematics. We claim that pure mathematics activities can also be accomplished by problems involving powers, which – despite their surprisingly elementary statements – can lead students to deep insights into the behaviour of numbers and the exciting possibilities in pure mathematics. Our goal in this paper is to demonstrate these alternative possibilities of pure mathematics by playing with powers, thus broadening the spectrum of the students.

2 POWER PROBLEMS AND REMAINDERS: LEAD-INS TO NUMBER THEORY BASICS

Students encounter basic number theory concepts such as prime and composite numbers, the division algorithm, divisibility tests, notions of least common multiple and greatest common divisor and the fundamental theorem of arithmetic in the middle school grades (10-13 year olds). Unfortunately there is little or no follow up to these topics in the higher grades. One of the curricular flaws of viewing Calculus as the pinnacle of the students' secondary schooling experience is working pre-dominantly over the set of real numbers with functions of continuous variables. The traditional sequence of Algebra-Geometry-Trigonometry and Analytic Geometry offers little opportunity to further develop elementary number theory notions that students have previously encountered. Problems involving powers offer a useful remedy to this unfortunate situation. Most of the extant books that include a treatment of problems involving powers and number theory are classified as "competition" books, thereby creating an "elitist" aura around such problems. We believe otherwise and invite teachers to make appropriate use of "power" tasks to convey to students the power of pure mathematics and the limitations of computing tools. An illustration of such a problem follows.

One can pose the question: What is the remainder when we divide 6^{131} by 215? Most calculators would not be able to handle 6^{131}, which immediately requires us to examine the problem with different conceptual tools. We can reflect on what remainders mean. We can go back to notions from the early grades and start basic computations such as:

What is the remainder when we divide 5 by 3? It's 2. That was easy.

What is the remainder when we divide $5 \times 8 = 40$ by 3? It's 1. Still easy!

Now what is the remainder when we divide $5 \times 8 \times 7 = 280$ by 3? We could laboriously perform the division by 3 and find that the remainder is 1.

Is there a lesson to be learned through this empirical work? Yes! The important mathematical phenomenon to communicate is that if we calculate the remainders on division by 3 of 5, 8 and 7 (which are 2,2, and 1 respectively) and multiply these individual remainders $2 \times 2 \times 1 = 4$ and divide this by 3, we get the same remainder when dividing 280 by 3. This phenomenon is in general called the Remainder Theorem, which can simply be stated as: the remainder of a product of numbers divided by a given number is equal to the remainder of the product of the remainders of the numbers (constituting the product) divided by this given number.

Now getting back to the original problem. We can employ the laws of exponents that our students so faithfully memorise and write 6^{131} as a product of numbers that leave a "nice" remainder when divided by 215. The best candidate is 6^3 since $6^3 = 216$ and 216 divided by 215 leaves a remainder of 1, and we can easily multiply 1's.

So $6^{131} = 6^3 \times 6^3 \times 6^3 \times 6^3\ldots(43 \text{ times}) \times 6^2$ divided by 215 leaves a product of remainders $1 \times 1 \times 1\times 1\ldots(43 \text{ times}) \times 36$, which is 36 and 36 divided by 215 leaves a remainder of 36. Done! This problem not only allowed us to make use of laws of exponents but also led to some insights into the theory of numbers.

Another nice problem involving powers is calculating the last digit of a given power, typically a huge number, such as 777^{777} or 2004^{2004}. Once again this allows us to make a nice foray into the basic behaviour of numbers. If we consider a simpler problem such as 1^{12345} we observe that the last digit is obviously 1 since we are simply multiplying a product of 1's. What if we had 11^{12345}, we can empirically verify that 11^1, 11^2, 11^3, 11^4…always end in 1. Students can be led to observe that the last digit is 1, and in general the last digit of any huge number such as 777^{777} is the product of the last digit multiplied by itself the given number of times. So the last digit of 777^{777} is the same as the last digit of 7^{777}. Now if we write out the powers of 7, we find a periodicity phenomenon in the last digits related to its powers. This is observed by listing out the sequence of the powers of 7: 7^1, 7^2, 7^3, 7^4, 7^5, 7^6, 7^7, 7^8, 7^9… which gives last digits 7,9,3,1,7,9,3,1,7…. So we can once again employ laws of exponents to rewrite 7^{777} as:

$7^{777} = (7^4) \times (7^4) \times (7^4) \times (7^4) \times\ldots(194 \text{ times}) \times 7^1$, so the last digit is 7.

Since we seem to be able to determine last digits of ridiculously large numbers, why not look at the problem of determining first digits.

3 STARTING DIGIT PROBLEMS: "EXCUSES" INTO COMBINATORICS AND ANALYSIS

Let's start with an existence problem. Is there an integral power of 2 that begins with 1999…in its decimal expansion? In other words the question is asking us to prove the existence of an integer 'n' such that $2^n = 1999\ldots$ without explicitly asking exactly what this power is. We can clearly assume that $n > 0$. When we raise 2 to any integer power, there are 9 obvious choices for the first digit since we naturally exclude zero, and then there are 10 choices for each digit after that. So there are $9\cdot 10\cdot10\cdot10 = 9000$ possible ways of listing the first four digits. Since $n > 0$, we have no restrictions on the number of values we can generate, and we can easily generate more than 9000 values for 2^n. By the pigeonhole principle, some powers of 2 have to begin with the same four-digit string but we are still not sure that one of those starting strings really equals 1999. This is a subtle question with an intriguing relation to irrationality and we postpone it for a second to mention something simpler.

The pigeonhole (or Dirichlet) principle states that if we have "m" pigeons and "n" pigeonholes, where $m > n$, then some pigeonhole contains more than one pigeon. This seemingly obvious principle has wide ranging applicability in mathematics.

A nice problem that is easily solved via the pigeonhole principle and makes use of divisibility properties is to prove that there exists some power of 3 that ends in 001. This can easily be shown as follows. Suppose 3^m and 3^n (where $m > n > 1$) upon division by 1000 have the same remainder. (The existence of such m and n needs to be established by applying the pigeonhole principle and this is left as an exercise for the readers.) Then $3^m - 3^n = 3^n(3^{m-n} - 1)$ is divisible by 1000. Now 3^n and 1000 clearly have no common factors, which means 1000 has to divide the factor $(3^{m-n} - 1)$. This implies 3^{m-n} ends with 001.

Now let us come back to powers of 2, and to the question whether one of them begins, in decimal notation, with 1999. Maybe this string is too strange to appear at the beginning of the decimal notation of some power of 2? Let us try something simpler first. Consider a sequence a(n) consisting of the first digits of the consecutive powers of 2:

1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, ... Will 7 ever appear in this sequence?

This problem, in different versions, is well known in mathematical literature. The first mention is found in the famous book *Ordinary differential equations* (Arnold, 1978). It is usually accompanied by auxiliary facts or suggestions intended to make a solution more accessible. Nevertheless, we have not encountered any detailed solutions to this problem. We will remedy this unfortunate situation and in the process illustrate the rich mathematics that comes out of it. To start with - a desperate solution using a pencil and paper or another powerful computing tool will easily verify that

$$2^{46} = 70\ 368\ 744\ 177\ 664$$

Going further with this experiment we can see that 7 is the first digit of 56^{th}, 66^{th}, 76^{th}, 86^{th} and the 96^{th} power of 2 (but the first digit of 106^{th} power of 2 is 8, not 7). This empirical method, however, is clearly far from being mathematically elegant. We need a better solution, which would allow us to draw other conclusions. First of all, we must try to realise the meaning of the statement that 7 is the first digit of a number 2^n. The answer is simple: 7 is the first digit of 2^n if and only if for some natural k we have

$$7 \times 10^k < 2^n < 8 \times 10^k$$

We can get a simpler description of this condition if we take the decimal logarithms of both sides, something our students would be quite familiar with. This yields

$$k + \log(7) < n \log(2) < k + \log(8).$$

Since the decimal logarithms of 7 and 8 lie between 0 and 1, we conclude that k must be the integer part of the number $n \log(2)$, which leads to the following inequalities:

$$\log 7 < n \log 2 - [n \log 2] < \log 8$$

And now it suffices to bring together some known facts, which we invite the readers to verify.

Lemma 1. *The number* log (2) *is irrational.*

Lemma 2. *If a number x is irrational and $c(n) := nx - [nx]$, then for any a and b belonging to $[0,1]$ infinitely many members of the sequence $c(n)$ lie in the interval (a,b).*

We hope that readers will encourage their students to prove lemma 1, which is very easily proved by contradiction. Assuming this done, let us have a look at a proof of the second lemma and then examine their consequences.

Proof of Lemma 2. Observe first that all the members of the sequence $c(n)$ are different. Indeed, if $c(k)=c(m)$ for $k > m$, then $(k-m)\,x = [kx]-[mx]$. This is a contradiction, since the product of a non-zero integer $(k-m)$ and an irrational number x cannot be an integer.

Take now a positive integer n such that $1/n < b-a$. The numbers $c(1)$, $c(2)$, ..., $c(n+1)$ being all different and belonging to the interval [0,1], we infer by the pigeon-hole principle that for some i and s such that i and $(i + s)$ both lie between 1 and $(n+1)$ the following inequality holds:

$$0 < \varepsilon = |c(i) - c(i+s)| \leq 1/n < b - a \qquad (1)$$

Now wrap the real axis into a circumference **T** of perimeter 1 with a distinguished point 0. For two numbers a and b in [0,1] we denote by (a,b) the arc of **T** which corresponds to the interval (a,b) on the real axis.

Let f**: T→ T** be an anti-clockwise revolution by an angle of $2\pi x$ radians. Instead of watching the numbers $c(n)$ in the interval [0,1] we shall look at the images of the distinguished point 0 under iterations of f on **T**. After a moment of reflection we note that the length of an arc $(0,b(n))$, where $b(n) = n\,(0) = f \bullet f \bullet \ldots \bullet f(0)$ [$\bullet$ = composition of mappings] is equal to $c(n)$. Hence, due to (1) we know that the length of an arc between the points $b(i)$ and $b\,(i+s)$ is smaller than $b - a$. This means that the s^{th} iterate of f is a revolution by an angle of $2\pi\varepsilon$ radians; the direction of this revolution is of no importance to us. This obviously implies that infinitely many of the points $b(s)$, $b(2s)$, $b(3s)$, ... belong to the arc (a, b). Indeed, if we start from the fixed point 0 and "walk" for an infinitely long time along the circumference **T** in one and the same direction making steps of length ε, then infinitely many times we shall step into the arc (a,b), since its length $b - a$ is greater than the length of our step. This completes the proof.

Now, by applying Lemma 2 to $x = \log(2)$, $a = \log(7)$, $b = \log(8)$ we find that 7 is the first digit of infinitely many powers of 2. If we apply again Lemma 2 to numbers

$x = \log(2)$, $a = \log(77) - 1$, $b = \log(78) - 1$,

then because of the equalities $1 = [\log 77] = [\log 78]$, we conclude that the figure seven can even appear twice in the first two places of the decimal notation of a power of 2. Using an analogous argument we readily discover that any finite sequence of digits can appear at the beginning of a decimal notation of a power of 2, like 1234 or 567890, or (finally) 1999. If you really do not believe this last statement, we invite you to compute (say) 2^{9030} or 2^{11166} . At the end of this article we exhibit some important historical dates compared with corresponding powers of 2 - to satisfy the skeptics. We can further deduce the following corollary:

Corollary. *If an integer $p > 1$ is not an integer power of* 10, *then any sequence of digits can appear at the beginning of the decimal notation of n-th power of p for some n.*

So the question again is why does 7 not appear among the first members of the sequence we introduced at the very beginning? Why does this deceitful sequence pretend to be periodic? The reason is simple. The number

$\log(2)=0.3010299956...$ can be very well approximated by the rational number 0.3, and for all rational x the sequence

$c(n) = nx - [nx]$ is periodic. In other words: $2^{10} = 1024$, which is quite close to 1000. And multiplication by 1000 just adds zeroes at the end; leading digits stay unchanged. This is why after seeing the first few members of the sequence $a(n)$ we come to an erroneous conclusion that the sequence has period 10 and 7 is not a member of it, while 8 appears quite often. To see the first 7 one has to look at the first 6 in $a(n)$ (i.e., the first digit of $64 = 2^6$), and then wait until the cumulative effects of small perturbation $24 = 2^{10} - 1000$ do their job.

In 1910 Waclaw Sierpinski, Hermann Weyl and P.Bohl proved independently of one another that for every irrational x the sequence $c(n) = nx - [nx]$ is equidistributed over the interval [0,1] (Arnold & Avez, 1968). More precisely, if we take arbitrary a and b (with $a < b$) from [0,1], and let

$k(n, a, b)$ denote the number of elements of the set

$\{c(i) : 1 \le i \le n, c(i) \in (a, b)\}$ then we obtain:

$$\lim_{n\to\infty} k(n,a,b)/n = b - a$$

Speaking in more illustrative terms, this theorem states that if we walk along a circle of circumference 1 unit, taking steps of irrational length, then we step on each hole with a frequency which is directly proportional to the size of the hole! Let's translate this fact into our language of powers of 2. Let $a(7,n)$ and $a(8,n)$ be the number of sevens and eights, correspondingly, among the first n members of the sequence $a(n)$. By the last formula, we have

$$\lim_{n\to\infty} a(7,n)/n = \log 8 - \log 7$$

$\lim_{n\to\infty} a(8,n)/n = \log 9 - \log 8$, so consequently

$$\lim_{n\to\infty} a(7,n)/a(8,n) = (\log 8 - \log 7)/(\log 9 - \log 8) = 1.1337.. > 1$$

This means that by looking at sufficiently long initial fragments of the sequence $a(n)$ we will see slightly more sevens than eights. This result of Bohl, Sierpinski and Weyl, which we previously mentioned and our little fact on 7's and 8's are in fact simple consequences of a very general and deep theorem in ergodic theory due to G.D.Birkhoff (Cornfeld, Fomin and Sinai, 1982), which the interested reader is urged to pursue.

An online reference with the specifics of this theorem is found in MathWorld--A Wolfram Web Resource.
http://mathworld.wolfram.com/BirkhoffsErgodicTheorem.html

We conclude this section with a little problem for our readers.

Problem: For what n does the number 2^n have four consecutive sevens at the beginning? What about five sevens? How can we estimate from above the least n such that the decimal notation of 2^n begins with 2004 consecutive sevens?

4 FUNCTIONAL PROBLEMS: GETTING DEEPER

If we "play" with the laws of powers (exponents), then we observe that $a^{x+y} = a^x \cdot a^y$. Let us assume that $a > 0$. The question can be posed generally as: Suppose f: R→ R, what are all the other solutions to $f(x + y) = f(x) \cdot f(y)$? Another good problem arises from observing that

$\log(xy) = \log x + \log y$. So what are all the other solutions to $f(x \cdot y) = f(x) + f(y)$? A classic related problem is of course to find all functions that satisfy the Cauchy functional equation:

$f(x+y) = f(x) + f(y)$. In answering these problems one gets into a deep investigation of functional equations, something that teachers can use as an extended project for the motivated and bright students.

5 SOME UNUSUAL POWERS OF 2

In order to satisfy the skeptics of the unusual property of powers of 2 we list some powers of 2 that connect with a (biased) sample of important historical dates.

The baptism of Poland	966	$2^{568} = 9.66... \times 10^{170}$
The Battle of Hastings	1066	$2^{5561} = 1.066... \times 10^{1674}$
Columbus discovers America	1492	$2^{3761} = 1.492... \times 10^{1132}$
The founding of Harvard University	1636	$2^{9528} = 1.636... \times 10^{2868}$
Cromwell's death	1658	$2^{3223} = 1.658... \times 10^{970}$
The founding of Royal Society	1660	$2^{4874} = 1.660... \times 10^{1467}$
New Amsterdam changes name to New York	1664	$2^{6040} = 1.664... \times 10^{1818}$
First edition of Newton's "Principia"	1687	$2^{6143} = 1.687\ldots \times 10^{1849}$
Walpole becomes Britain's first Prime Minister	1721	$2^{10229} = 1.721\ldots \times 10^{3079}$
French Revolution	1789	$2^{9857} = 1.789\ldots \times 10^{2967}$
Waterloo	1815	$2^{931} = 1.815\ldots \times 10^{280}$
Beginning of World War II	1939	$2^{5522} = 1.939\ldots \times 10^{1662}$
End of World War II	1945	$2^{1931} = 1.945\ldots \times 10^{581}$

6 CONCLUSIONS

We hope to have conveyed to the readers the richness of pure mathematics present in problems involving powers. Playing with problems involving powers offers opportunities to make deep connections with topics in Number Theory, Combinatorics and Analysis and complements the Applied Mathematics and Statistics that students learn through the modelling approach that is presently gathering momentum. The first author used first and last digit problems similar to the ones mentioned in this paper with 14-year old pupils enrolled in an Algebra course. The pedagogical goal was to mediate "pure math" problem solving experiences and resulted in considerable student interest in the mysteries of the integers. Among other things, students realised the limitations of computing tools and understood the need to create/invent conceptual tools to tackle the problem. Other problems involving a particular phenomenon among the positive integers and resulting in the discovery of the pigeonhole principle also met with great success in the classroom (Sriraman, 2004a; 2004b).

In conclusion, our hope is that we never forget the aesthetic beauty inherent in pure mathematics activities and convey to our students that such activities have sustained the imagination of mathematicians and contributed to its growth from the very onset of its history. We feel that the image of a pure mathematician lying under a tree (apparently doing "nothing" to the untrained eye!) complements and balances the image of the diligent applied mathematician and scientist engrossed in making sense of the hubris and chaos of the real world. After all, what would the second person do if the first one really did nothing?

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REFERENCES

Arnold, V.I. (1978) *Ordinary differential equations* (translated from Russian by R.A. Silverman): Massachusetts: MIT Press.

Arnold, V.I., & Avez, A. (1968) *Ergodic Problems in Classical Mechanics*, Benjamin, New York.

Bourbaki, N. (1970). Théorie des Ensembles dela collection elements de Mathématique, Hermann, Paris.

Coleman, J. (1964). *Introduction to Mathematical Sociology*. New York: The Free Press.

Cornfeld, I., Fomin, S., and Sinai, Ya. G. (1982). *Ergodic Theory.* New York: Springer-Verlag.

Lesh, R & Doerr, H. (2003). Foundations of a models and modelling perspective on mathematics teaching, learning and problem solving. In R. Lesh and H. Doerr (Eds.), *Beyond Constructivism* (pp.3-34). New Jersey: Lawrence Erlbaum Associates.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*: Reston, VA: Author.

Sriraman, B. (2004a). Discovering a mathematical principle: The case of Matt. *Mathematics in School*,**33 No 2**, 25-31.

Sriraman, B. (2004b). Reflective abstraction, uniframes and the formulation of generalizations. *The Journal of Mathematical Behavior*.**23 No 2**, 207-224.

BIOGRAPHICAL NOTES

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